On the Differential Equations Connected with Hypersurfaces.

Dissertation

Presented to the Board of University Studies of the Johns Hopkins University for the Degree of Doctor of Philosophy

by

George Oscar James

Baltimore

1899

Contents

This work was done under the direction of Professor Craig and I do wish here to thank him for the many helpful suggestions he has given me and I wish to especially express my sincere appreciation of his kindness shown and of the interest he has taken in my work.

To Professor Ames I am grateful for the kindly way in which he has helped me in my work in physics. Here his advice has been invaluable both during my undergraduate and graduate courses.

In the course of this work certain partial differential equations of special forms have arisen and it is these which lend interest to the study and development of the theory of hypersurfaces.

The generalized Riccati Equations and the equations of the type of Laplace in three independent variables are intimately connected with the existence of a hypersurface corresponding to two given fundamental forms, and the generalization of the notion of conjugate lines and lines of curvature leads to important information concerning these latter.

Later I hope to again take up the study of these equations.

I

If we define a curved space of $n-1$ dimensions as a manifoldness of $n-1$ dimensions, which is contained in homaloidal (Euclidean) space of n dimensions and which can be moved about in this n-dimensional space without alteration of its linear elements, then the coordinates of a point on this curved space can be represented by n variables and a single equation connecting these will serve to determine the curved space. Again, m such equations will determine a curved space of $n-m$ dimensions $n-2$ a surface and

n — a twisted curve.

The theory of curves of multiple curvature in n-dimensional space is now developed to some extent and very interesting results appear in two papers by Brunel* and Pin-[illegible]**. In his thesis presented to this University in 895 Mr. C. C. Hodgson found the Serret-Frenet formulae for curves of triple curvature by the kinematical method of Darboux, and in a paper in the American Journal of Mathematics, Vol. IX, Professor Craig has by this method generalized a number of formulae relating to the theory of surfaces. Other references to articles on curved spaces will appear in this paper.

---

* Sur les propriétés métriques des courbes gauches dans un espace linéaire à n dimensions — Math. Annalen, Band 19.

** Sulle linee a triple curvatura delle varietà chiuse a n dimensioni. [illegible]

I shall confine myself, on, for the most part to homaloidal or Euclidian space of four dimensions and shall adopt the nomenclature used by Poincaré in his memoir* sur les Résidus des Intégrales Doubles and employed by Professor Craig in the paper referred to above. Four dimensional homaloidal space will be termed *hyperspace* and a single relation between the coordinates of a point in hyperspace will define a *hypersurface*, two such relations a surface and three a line.

Consider then the hypersurface

$$F(xyzw) = 0$$

of linear element

* Acta Math. t. 9 pg 3 &c.

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

Express the coordinates in terms of three independent parameters by the formulae

$x = x(tuv)$, $y = y(tuv)$, $z = z(tuv)$, $w = w(tuv)$

which together with their first second and third partial derivatives, are suppose uniform finite and continuous throughout the region of variation of t.

The linear element then takes the form

$$ds^2 = \mathcal{E}_{11} dt^2 + \mathcal{E}_{22} du^2 + \mathcal{E}_{33} dv^2 + 2\mathcal{E}_{12} dt\,du + 2\mathcal{E}_{13} dt\,dv + 2\mathcal{E}_{23} du\,dv$$

where

$$\mathcal{E}_{11} = \sum\left(\frac{\partial x}{\partial t}\right)^2, \quad \mathcal{E}_{22} = \sum\left(\frac{\partial x}{\partial u}\right)^2, \quad \mathcal{E}_{33} = \sum\left(\frac{\partial x}{\partial v}\right)^2$$

$$\mathcal{E}_{12} = \sum\frac{\partial x}{\partial t}\frac{\partial x}{\partial u}, \quad \mathcal{E}_{13} = \sum\frac{\partial x}{\partial t}\frac{\partial x}{\partial v}, \quad \mathcal{E}_{23} = \sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}$$

The study of this ternary differential quadratic form will be the basis of this paper, and after the analogy of the theory of surfaces I shall call it the

first fundamental form of the hypersurface.

Differential quadratic forms in n variables have been studied by many mathematicians notably by Christoffel*, Lipschitz**, Voss† and Ricci†† and some applications of the theory of curved spaces have been made by Italian geometers references to which will appear later.

" A differential quadratic form in n variables

$$f = \sum_{1}^{n} a_{rs}\, dx_r\, dx_s \qquad (1)$$

is said to be reducible‡ whenever it can be put indentically in the form

*Crelle's Journal Band 70

** " " Bände 71, 72, 74, 75, 81.

† Math. Annalen Band 16.

†† Annali di Matematica serie II, vol. : Principii di una teoria delle forme differenziali quadratiche.

‡ Ricci, Loc. Cit. pag. 130.

$$f = \sum_{lm}^{n-1} a_{lm}\, du_l\, du_m$$

where the variables $u_1 \cdots u_{n-1}$ are functions of $x_1 \cdots x_n$ and the coefficients a_{lm} are functions of $u_1 \cdots u_{n-1}$.

The necessary and sufficient conditions that f be reducible are*

$$a = 0$$

$$\sum_i \alpha_{pi} \left[{rs \atop i}\right] = 0$$

where a is the discriminant of f, α_{rs} the algebraic complement of a_{rs} in a and

$$\left[{rs \atop i}\right] = \frac{1}{2}\left(\frac{\partial a_{is}}{\partial x_r} + \frac{\partial a_{ir}}{\partial x_s} - \frac{\partial a_{rs}}{\partial x_i}\right)$$

Suppose now that f is irreducible. In this case Schläfli** has shown that it can be deduced from the form

$$ds^2 = \sum_t^{n+h} dy_t^2 \qquad (2$$

where $0 \leq h \leq \frac{n(n-1)}{2}$

* Ricci loc. cit. page 142.
** Annali di Matematica. Series I. Vol. 5.

is known as the class* of the form f.
Consider the hypersurface

$x = x(t,u,v),\ y = y(tuv),\ z = z(tuv),\ w = w(tuv)$

where x, y, z, w fulfil the conditions prescribed above. If one of the parameters, t say, be held constant we get a coordinate surface which I shall call the t surface. If two of them, u and v say, be held constant we get a coordinate line which I shall call the t line. It is to be noted that the t surface is the surface

$$t = \text{constant}$$

while the t line is the line

$$u = \text{const.},\quad v = \text{const.}$$

Jordan** has shown that given the two right lines through the origin

$$\frac{y_1}{a_1} = - - - = \frac{y_n}{a_n}$$

* Ricci, Loc. Cit., pag. 142.
** Bulletin de la Société Mathématique de France, t. III, pp. 103 —

$$\frac{x_1}{a_1} = - - - - = \frac{x_n}{a_n}$$

in an n dimensional space the expression

$$\cos^2\theta = \frac{\Sigma(b_r a_r)^2}{\Sigma b_r^2 \Sigma a_r^2}$$

is an invariant under orthogonal substitution and θ is known as the angle between the two lines. p linear equations in the coordinates define a linear space of n-p dimensions and if we have two linear spaces one of p and the other of q dimensions we can calculate by means of this formula the angle between any two lines one in each space. The expression thus obtained has its maxima and minima invariant under orthogonal transformation and these define the angles between the two spaces. The num

ber of angles between two such planes is the least of the numbers p, q, $n-p, n-q$. Hence in hyperspace two hyperplanes make a single angle and if the hyperplanes are written

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 + \alpha = 0$$

$$b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4 + b = 0$$

the angle is given by

$$\cos^2\vartheta = \frac{\sum (a_r b_r)^2}{\sum a_r^2 \sum b_r^2}$$

Write ω_{12} = angle between t-line and u-line

ω_{13} = " " t-line " v-line

ω_{23} = " " u-line " v-line

Denoting by $\cos(tx)$ the cosine of the angle between the t-line and the x axis we have

$$\cos\omega_{12} = \sum \cos(tx)\cos(ux) = \frac{1}{\sqrt{E_{11}E_{22}}} \sum \frac{\partial x}{\partial t}\frac{\partial x}{\partial u}$$

$$= \frac{E_{12}}{\sqrt{E_{11}E_{22}}}$$

Similarly $\cos\omega_{13} = \frac{E_{13}}{\sqrt{E_{11}E_{33}}}$, $\cos\omega_{23} = \frac{E_{23}}{\sqrt{E_{22}E_{33}}}$

Hence the necessary and sufficient conditions that the parametric lines form a triply orthogonal system are

$$E_{12} = E_{13} = E_{23} = 0$$

and the linear element becomes

$$ds^2 = E_{11}dt^2 + E_{22}du^2 + E_{33}dv^2$$

Since the parametric surfaces, taken in pairs form six angles the ordinary meaning of triply orthogonal system of surfaces is not immediately extensible to hyperspace but in the case where we have a triply orthogonal system of parametric lines these six angles all all become right angles and we have a true triply orthogonal system. This is easily seen as follows.

The tangent plane to the v surf-

are at the point $M(xyzw)$ is given by the t and u lines, that is by the following direction cosines

$$P\begin{cases} \frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}, & \frac{1}{\sqrt{E_{11}}}\frac{\partial y}{\partial t}, & \frac{1}{\sqrt{E_{11}}}\frac{\partial z}{\partial t}, & \frac{1}{\sqrt{E_{11}}}\frac{\partial w}{\partial t} \\ \frac{1}{\sqrt{E_{22}}}\frac{\partial x}{\partial u}, & \frac{1}{\sqrt{E_{22}}}\frac{\partial y}{\partial u}, & \frac{1}{\sqrt{E_{22}}}\frac{\partial z}{\partial u}, & \frac{1}{\sqrt{E_{22}}}\frac{\partial w}{\partial u} \end{cases}$$

Similarly the tangent plane of the u surfaces is given by

$$\Pi\begin{cases} \frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}, & \frac{1}{\sqrt{E_{11}}}\frac{\partial y}{\partial t}, & \frac{1}{\sqrt{E_{11}}}\frac{\partial z}{\partial t}, & \frac{1}{\sqrt{E_{11}}}\frac{\partial w}{\partial t} \\ \frac{1}{\sqrt{E_{33}}}\frac{\partial x}{\partial v}, & \frac{1}{\sqrt{E_{33}}}\frac{\partial y}{\partial v}, & \frac{1}{\sqrt{E_{33}}}\frac{\partial z}{\partial v}, & \frac{1}{\sqrt{E_{33}}}\frac{\partial w}{\partial v} \end{cases}$$

Applying Jordan's method the two angles between these two planes P and Π are given by

$$\begin{vmatrix} \cos\theta & \cos\theta\frac{E_{11}}{\sqrt{E_{11}E_{12}}} & 1 & \frac{E_{13}}{\sqrt{E_{11}E_{33}}} \\ \cos\theta\frac{E_{11}}{\sqrt{E_{11}E_{22}}} & \cos\theta & \frac{E_{12}}{\sqrt{E_{11}E_{22}}} & \frac{E_{23}}{\sqrt{E_{11}E_{33}}} \\ 1 & \frac{E_{12}}{\sqrt{E_{11}E_{12}}} & \cos\theta & \cos\theta\frac{E_{13}}{\sqrt{E_{11}E_{33}}} \\ \frac{E_{13}}{\sqrt{E_{11}E_{33}}} & \frac{E_{23}}{\sqrt{E_{22}E_{33}}} & \cos\theta\frac{E_{13}}{\sqrt{E_{11}E_{33}}} & \cos\theta \end{vmatrix} = 0$$

and it is easily seen that if

$$\Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0$$

the three angles defined by these equations are right angles and similarly for the other pairs of coordinate surfaces.

Let C be a line traced on the hypersurface making angles $\vartheta_1, \vartheta_2, \vartheta_3$ with the coordinate lines. Then

$$\cos\vartheta_1 = \sum \cos(t\,x)\cos(c\,x) = \frac{1}{\sqrt{\Sigma_{11}}}\left(\Sigma_{11}\frac{dt}{ds} + \Sigma_{12}\frac{du}{ds} + \Sigma_{13}\frac{dv}{ds}\right)$$

$$\cos\vartheta_2 = \frac{1}{\sqrt{\Sigma_{22}}}\left(\Sigma_{21}\frac{dt}{ds} + \Sigma_{22}\frac{du}{ds} + \Sigma_{23}\frac{dv}{ds}\right), \quad \cos\vartheta_3 = \frac{1}{\sqrt{\Sigma_{33}}}\left(\Sigma_{31}\frac{dt}{ds} + \Sigma_{32}\frac{du}{ds} + \Sigma_{33}\frac{dv}{ds}\right)$$

Let C' be a second line and denote its increments by δ. The angle between C and C′ is then

$$\cos\vartheta = \sum \cos(c\,x)\cos(c'\,x)$$

$$= \Sigma_{11}\frac{dt}{ds}\frac{\delta t}{\delta s} + \Sigma_{22}\frac{du}{ds}\frac{\delta u}{\delta s} + \Sigma_{33}\frac{\delta v}{\delta s}$$

$$+ \Sigma_{12}\left(\frac{dt}{ds}\frac{\delta u}{\delta s} + \frac{\delta t}{\delta s}\frac{du}{ds}\right) + \Sigma_{13}\left(\frac{dt}{ds}\frac{\delta v}{\delta s} + \frac{\delta t}{\delta s}\frac{dv}{ds}\right) + \Sigma_{23}\left(\frac{du}{ds}\frac{\delta v}{\delta s} + \frac{\delta u}{\delta s}\frac{dv}{ds}\right)$$

Hence the condition that C and C′ be orthogonal is that the right hand member of this equation vanish.

Christoffel symbols

Writing $x_1 = t,\ x_2 = u,\ x_3 = v,\ a_{ij} = \mathcal{E}_{ij}$ and applying the formula defining the symbols of the first kind

$$\left[{i\,k \atop l}\right] = \frac{1}{2}\left(\frac{\partial a_{il}}{\partial x_k} + \frac{\partial a_{kl}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_l}\right)$$

we have the following eighteen symbols of the first kind of three indices for the case of four variables.

$$\left[{1\,1 \atop 1}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{11}}{\partial t};\quad \left[{1\,2 \atop 1}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{11}}{\partial u};\quad \left[{1\,3 \atop 1}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{11}}{\partial v};$$

$$\left[{2\,2 \atop 1}\right] = \frac{\partial \mathcal{E}_{12}}{\partial u} - \frac{1}{2}\frac{\partial \mathcal{E}_{22}}{\partial t};\quad \left[{2\,3 \atop 1}\right] = \frac{1}{2}\left(\frac{\partial \mathcal{E}_{12}}{\partial v} + \frac{\partial \mathcal{E}_{13}}{\partial u} - \frac{\partial \mathcal{E}_{23}}{\partial t}\right);$$

$$\left[{3\,3 \atop 1}\right] = \frac{\partial \mathcal{E}_{13}}{\partial v} - \frac{1}{2}\frac{\partial \mathcal{E}_{33}}{\partial v};\quad \left[{1\,1 \atop 2}\right] = \frac{\partial \mathcal{E}_{12}}{\partial t} - \frac{1}{2}\frac{\partial \mathcal{E}_{11}}{\partial u};$$

$$\left[{1\,2 \atop 2}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{22}}{\partial t};\quad \left[{1\,3 \atop 2}\right] = \frac{1}{2}\left(\frac{\partial \mathcal{E}_{12}}{\partial v} + \frac{\partial \mathcal{E}_{23}}{\partial t} - \frac{\partial \mathcal{E}_{13}}{\partial u}\right);\quad \left[{2\,2 \atop 2}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{22}}{\partial u};$$

$$\left[{2\,3 \atop 2}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{22}}{\partial v};\quad \left[{3\,3 \atop 2}\right] = \frac{\partial \mathcal{E}_{23}}{\partial v} - \frac{1}{2}\frac{\partial \mathcal{E}_{33}}{\partial u};\quad \left[{1\,1 \atop 3}\right] = \frac{\partial \mathcal{E}_{13}}{\partial t} - \frac{1}{2}\frac{\partial \mathcal{E}_{11}}{\partial v};$$

$$\left[{1\,2 \atop 3}\right] = \frac{1}{2}\left(\frac{\partial \mathcal{E}_{13}}{\partial u} + \frac{\partial \mathcal{E}_{23}}{\partial t} - \frac{\partial \mathcal{E}_{12}}{\partial v}\right);\quad \left[{1\,3 \atop 3}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{33}}{\partial t};$$

$$\left[{2\,2 \atop 3}\right] = \frac{\partial \mathcal{E}_{23}}{\partial u} - \frac{1}{2}\frac{\partial \mathcal{E}_{22}}{\partial v},\quad \left[{2\,3 \atop 3}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{33}}{\partial u};\quad \left[{3\,3 \atop 3}\right] = \frac{1}{2}\frac{\partial \mathcal{E}_{33}}{\partial v}.$$

The symbols of the second kind of three indices will be useful later when the hypersurface is referred to a triple orthogonal system and I shall calculate them for that case. Using

$$\varepsilon_{12}=\varepsilon_{13}=\varepsilon_{23}=0$$

in the symbols of the first kind these are given at once and to get those of the second kind it is sufficient to observe that

$$\tau_{11}=\frac{1}{\varepsilon_{11}},\quad \tau_{22}=\frac{1}{\varepsilon_{22}},\quad \tau_{33}=\frac{1}{\varepsilon_{33}},\quad \tau_{il}=0\ (i\neq l)$$

where τ_{ij} is the algebraic complement of a_{ij} in a divided by a itself.

Computing then the symbols of the second kind from the formula

$$\left\{{ik \atop r}\right\}=\sum_l \tau_{rl}\left[{ik \atop l}\right]$$

we have

$$\left\{{11 \atop 1}\right\}=\frac{1}{\sqrt{\varepsilon_{11}}}\frac{\partial\sqrt{\varepsilon_{11}}}{\partial u_1},\quad \left\{{12 \atop 1}\right\}=\frac{1}{\sqrt{\varepsilon_{11}}}\frac{\partial\sqrt{\varepsilon_{11}}}{\partial u_2},\quad \left\{{13 \atop 1}\right\}=\frac{1}{\sqrt{\varepsilon_{11}}}\frac{\partial\sqrt{\varepsilon_{11}}}{\partial u_3},$$

$$\left\{{22 \atop 1}\right\}=-\frac{\sqrt{\varepsilon_{22}}}{\varepsilon_{11}}\frac{\partial\sqrt{\varepsilon_{22}}}{\partial u_1};\quad \left\{{23 \atop 1}\right\}=0;\quad \left\{{33 \atop 1}\right\}=-\frac{\sqrt{\varepsilon_{33}}}{\varepsilon_{11}}\frac{\partial\sqrt{\varepsilon_{33}}}{\partial u_1},$$

$$\left\{{11 \atop 2}\right\} = -\frac{\sqrt{E_{11}}}{E_{22}}\frac{\partial\sqrt{E_{11}}}{\partial u}, \quad \left\{{12 \atop 2}\right\} = \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial t}, \quad \left\{{13 \atop 2}\right\} = 0;$$

$$\left\{{22 \atop 2}\right\} = \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial u}, \quad \left\{{23 \atop 2}\right\} = \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial v}; \quad \left\{{33 \atop 2}\right\} = -\frac{\sqrt{E_{33}}}{E_{22}}\frac{\partial\sqrt{E_{33}}}{\partial u}$$

$$\left\{{11 \atop 3}\right\} = -\frac{\sqrt{E_{11}}}{E_{33}}\frac{\partial\sqrt{E_{11}}}{\partial v}; \quad \left\{{12 \atop 3}\right\} = 0, \quad \left\{{13 \atop 3}\right\} = \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{33}}}{\partial t}.$$

$$\left\{{22 \atop 3}\right\} = -\frac{\sqrt{E_{22}}}{E_{33}}\frac{\partial\sqrt{E_{22}}}{\partial v}, \quad \left\{{23 \atop 3}\right\} = \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{33}}}{\partial u}; \quad \left\{{33 \atop 3}\right\} = \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{33}}}{\partial v}.$$

Denote now the discriminant of the linear element by a when expressed in the a_{ij} and by A when expressed in the E_{ij} and calculate the differential parameters $\Delta\varphi$, $\Delta_2\varphi$, $\Delta(\varphi\psi)$ from the formulae

$$\Delta\varphi = \sum_{1}^{3}{}_{rs} \, A_{rs}\frac{\partial\varphi}{\partial x_r}\frac{\partial\varphi}{\partial x_s}$$

$$\Delta_2\varphi = \frac{1}{\sqrt{a}}\sum_{1}^{3}{}_{s}\frac{\partial}{\partial x_s}\sum_{1}^{3}{}_{r} A_{rs}\sqrt{a}\frac{\partial\varphi}{\partial x_r}$$

$$\Delta(\varphi\psi) = \sum_{1}^{3}{}_{rs} A_{rs}\frac{\partial\varphi}{\partial x_r}\frac{\partial\varphi}{\partial x_s}$$

The necessary and sufficient condition that the two surfaces

$\varphi = \text{const.} \quad \psi = \text{const.}$

orthogonal is thus expressed by*

$$\Delta(\varphi\,\psi') = 0$$

II

The tangent hyperplane is determined by any three lines lying in it and in particular by the tangents to the three coordinate lines. Defining then the normal to the hypersurface at any point M as the line perpendicular to the hypertangent plane at M, we have to express that the line whose direction cosines are X, Y, Z, W is orthogonal to the three coordinate lines. We have then

$$\sum X\frac{\partial x}{\partial t} = 0 \quad \sum X\frac{\partial x}{\partial u} = 0 \quad \sum X\frac{\partial x}{\partial v} = 0$$

These give for the direction cosines of the normal

$$X = \frac{1}{\sqrt{\Delta}}\frac{\partial(yzw)}{\partial(tuv)}, \quad Y = \frac{1}{\sqrt{\Delta}}\frac{\partial(zwx)}{\partial(tuv)}, \quad Z = \frac{1}{\sqrt{\Delta}}\frac{\partial(wxy)}{\partial(tuv)}, \quad W = \frac{1}{\sqrt{\Delta}}\frac{\partial(xyz)}{\partial(tuv)} \qquad (1)$$

* Pedora – ... coordinate curvilinee – Rend ... [illegible]

Introduce now the second differential quadratic form

$$\varphi = -\sum dX\,dx$$

$$\equiv a_{11}\,dt^2 + a_{22}\,du^2 + a_{33}\,dv^2 + 2a_{12}\,dt\,du + 2a_{13}\,dt\,dv + 2a_{23}\,du\,dv \qquad (2$$

We at once find

$$\left.\begin{aligned}
a_{11} &= \sum X \frac{\partial^2 x}{\partial t^2} = -\sum \frac{\partial X}{\partial t}\frac{\partial x}{\partial t} \\
a_{22} &= \sum X \frac{\partial^2 x}{\partial u^2} = -\sum \frac{\partial X}{\partial u}\frac{\partial x}{\partial u} \\
a_{33} &= \sum X \frac{\partial^2 x}{\partial v^2} = -\sum \frac{\partial X}{\partial v}\frac{\partial x}{\partial v} \\
a_{12} &= \sum X \frac{\partial^2 x}{\partial t\,\partial u} = -\sum \frac{\partial X}{\partial t}\frac{\partial x}{\partial u} = -\sum \frac{\partial X}{\partial u}\frac{\partial x}{\partial t} \\
a_{13} &= \sum X \frac{\partial^2 x}{\partial t\,\partial v} = -\sum \frac{\partial X}{\partial t}\frac{\partial x}{\partial v} = -\sum \frac{\partial X}{\partial v}\frac{\partial x}{\partial t} \\
a_{23} &= \sum X \frac{\partial^2 x}{\partial u\,\partial v} = -\sum \frac{\partial X}{\partial u}\frac{\partial x}{\partial v} = -\sum \frac{\partial X}{\partial v}\frac{\partial x}{\partial u}
\end{aligned}\right\} \qquad (3)$$

By means of the expressions (1) these can be written

$$a_{11} = \frac{1}{\sqrt{\Delta}} \begin{vmatrix} \frac{\partial^2 x}{\partial t^2} & \frac{\partial^2 y}{\partial t^2} & \frac{\partial^2 z}{\partial t^2} & \frac{\partial^2 w}{\partial t^2} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} & \frac{\partial w}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} & \frac{\partial w}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} & \frac{\partial w}{\partial v} \end{vmatrix} \qquad (3^*)$$

with similar expressions for the remaining five.

If now A, B, C, D be any four functions whatever of $t\ u\ v$ then $\alpha, \beta, \gamma, \delta$ can be so determined that the following four equations are satisfied,

$$\left.\begin{aligned} A &= \alpha \frac{\partial x}{\partial t} + \beta \frac{\partial x}{\partial u} + \gamma \frac{\partial x}{\partial v} + \delta X \\ B &= \alpha \frac{\partial y}{\partial t} + \beta \frac{\partial y}{\partial u} + \gamma \frac{\partial y}{\partial v} + \delta Y \\ C &= \alpha \frac{\partial z}{\partial t} + \beta \frac{\partial z}{\partial u} + \gamma \frac{\partial z}{\partial v} + \delta Z \\ D &= \alpha \frac{\partial w}{\partial t} + \beta \frac{\partial w}{\partial u} + \gamma \frac{\partial w}{\partial v} + \delta W \end{aligned}\right\} \quad (a)$$

since the determinant of the system equals $\sqrt{\Delta}$ and is therefore not zero.

By exactly the same reasoning employed in the case of two parameters+ we arrive at the following system of equations satisfied by the four coordinates x, y, z, w.

$$\frac{\partial^2 \theta}{\partial t^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \frac{\partial \theta}{\partial t} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \frac{\partial \theta}{\partial u} + \begin{Bmatrix} 11 \\ 3 \end{Bmatrix} \frac{\partial \theta}{\partial v} + D_{11} X$$

$$\frac{\partial^2 \theta}{\partial t \partial u} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} \frac{\partial \theta}{\partial t} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} \frac{\partial \theta}{\partial u} + \begin{Bmatrix} 12 \\ 3 \end{Bmatrix} \frac{\partial \theta}{\partial v} + D_{12} X$$

+ Bianchi – Lezioni di Geometria Differenziale – §. 57.

$$\left.\begin{aligned}
\frac{\partial^2\theta}{\partial t\,\partial v} &= \left\{{13 \atop 1}\right\}\frac{\partial\theta}{\partial t} + \left\{{13 \atop 2}\right\}\frac{\partial\theta}{\partial u} + \left\{{13 \atop 3}\right\}\frac{\partial\theta}{\partial v} + D_{13}X \\
\frac{\partial^2\theta}{\partial u^2} &= \left\{{22 \atop 1}\right\}\frac{\partial\theta}{\partial t} + \left\{{22 \atop 2}\right\}\frac{\partial\theta}{\partial u} + \left\{{22 \atop 3}\right\}\frac{\partial\theta}{\partial v} + D_{22}X \\
\frac{\partial^2\theta}{\partial u\,\partial v} &= \left\{{23 \atop 1}\right\}\frac{\partial\theta}{\partial t} + \left\{{23 \atop 2}\right\}\frac{\partial\theta}{\partial u} + \left\{{23 \atop 3}\right\}\frac{\partial\theta}{\partial v} + D_{23}X \\
\frac{\partial^2\theta}{\partial v^2} &= \left\{{33 \atop 1}\right\}\frac{\partial\theta}{\partial t} + \left\{{33 \atop 2}\right\}\frac{\partial\theta}{\partial u} + \left\{{33 \atop 3}\right\}\frac{\partial\theta}{\partial v} + D_{33}X
\end{aligned}\right\} \qquad (I)$$

and the three equations satisfied by the direction cosines of the normal

$$\left.\begin{aligned}
\frac{\partial\Theta}{\partial t} &= -\frac{1}{\Delta}|D_{11}^1|\frac{\partial\theta}{\partial t} - \frac{1}{\Delta}|D_{11}^2|\frac{\partial\theta}{\partial u} - \frac{1}{\Delta}|D_{11}^3|\frac{\partial\theta}{\partial v} \\
\frac{\partial\Theta}{\partial u} &= -\frac{1}{\Delta}|D_{21}^1|\frac{\partial\theta}{\partial t} - \frac{1}{\Delta}|D_{21}^2|\frac{\partial\theta}{\partial u} - \frac{1}{\Delta}|D_{21}^3|\frac{\partial\theta}{\partial v} \\
\frac{\partial\Theta}{\partial v} &= -\frac{1}{\Delta}|D_{31}^1|\frac{\partial\theta}{\partial t} - \frac{1}{\Delta}|D_{31}^2|\frac{\partial\theta}{\partial u} - \frac{1}{\Delta}|D_{31}^3|\frac{\partial\theta}{\partial v}
\end{aligned}\right\} \qquad (II)$$

where

$$|D_{i1}^1| \equiv \begin{vmatrix} D_{i1} & E_{12} & E_{13} \\ D_{i2} & E_{22} & E_{23} \\ D_{i3} & E_{32} & E_{33} \end{vmatrix} \qquad (i = 1, 2, 3)$$

the column of D_i standing in the first, second or third place according to the upper index, hence in the first place in this case.

The relations connecting the coefficients of the first and second forms can now be found by writing the conditions that the two systems of partial differential equations (I) and (II) be integrable.

Putting as usual x_1, x_2, x_3 for the parameters t, u, v the conditions of integrability are for (I)

$$\frac{\partial}{\partial x_t}\left(\frac{\partial^2 \theta}{\partial x_r \partial x_s}\right) = \frac{\partial}{\partial x_r}\left(\frac{\partial^2 \theta}{\partial x_s \partial x_t}\right)$$

$$t, r, s = 1, 2, 3, \quad t \neq r, \quad t \neq s.$$

These give

$$\sum_i \left[\begin{Bmatrix} rs \\ i \end{Bmatrix}\frac{\partial^2 \theta}{\partial x_i \partial x_t} + \frac{\partial}{\partial x_t}\begin{Bmatrix} rs \\ i \end{Bmatrix}\frac{\partial \theta}{\partial x_i} + \Theta\frac{\partial \lambda_{rs}}{\partial x_t} + \lambda_{rs}\frac{\partial \Theta}{\partial x_t}\right] =$$

$$\sum_i \left[\begin{Bmatrix} st \\ i \end{Bmatrix}\frac{\partial^2 \theta}{\partial x_i \partial x_r} + \frac{\partial}{\partial x_r}\begin{Bmatrix} st \\ i \end{Bmatrix}\frac{\partial \theta}{\partial x_i} + \Theta\frac{\partial \lambda_{st}}{\partial x_r} + \lambda_{st}\frac{\partial \Theta}{\partial x_r}\right]$$

Substituting for $\frac{\partial^2 \theta}{\partial x_r \partial x_s}$ and $\frac{\partial \Theta}{\partial x_t}$, from equations (I) and (II) we have

$$\sum_{ij}\Big[\Big(\begin{Bmatrix} rs \\ i \end{Bmatrix}\begin{Bmatrix} it \\ j \end{Bmatrix} - \begin{Bmatrix} st \\ i \end{Bmatrix}\begin{Bmatrix} ir \\ j \end{Bmatrix} + \frac{\partial}{\partial x_t}\begin{Bmatrix} rs \\ j \end{Bmatrix} - \frac{\partial}{\partial x_r}\begin{Bmatrix} st \\ j \end{Bmatrix} -$$

$$\frac{\mathfrak{D}_{rs}}{\Delta}|\mathfrak{D}^{j}_{t,}| + \frac{\mathfrak{D}_{st}}{\Delta}|\mathfrak{D}^{j}_{r,}|\Big)\frac{\partial\theta}{\partial x_j} +$$

$$\Big(\frac{\partial \mathfrak{D}_{rs}}{\partial x_t} - \frac{\partial \mathfrak{D}_{st}}{\partial x_r} + \mathfrak{D}_{jt}\begin{Bmatrix} rs \\ j \end{Bmatrix} - \mathfrak{D}_{jr}\begin{Bmatrix} st \\ j \end{Bmatrix}\Big)\Theta\Big] = 0 \qquad (6)$$

Substituting x, y, z, w successively for t we have a system of four equations whose determinant

$$\begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & X \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & Y \\ \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & Z \\ \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} & W \end{vmatrix} = \sqrt{\Delta}$$

does not vanish, and hence the coefficients must separately vanish. Introducing here the ~~Christoffel~~ Riemann symbols of four indices

$$\{sj, rt\} = \frac{\partial}{\partial x_t}\begin{Bmatrix} rs \\ j \end{Bmatrix} - \frac{\partial}{\partial x_r}\begin{Bmatrix} st \\ j \end{Bmatrix} + \sum_i\Big(\begin{Bmatrix} rs \\ i \end{Bmatrix}\begin{Bmatrix} it \\ j \end{Bmatrix} - \begin{Bmatrix} st \\ i \end{Bmatrix}\begin{Bmatrix} ir \\ j \end{Bmatrix}\Big)$$

we have the equations,

$$\{sj\,rt\} - \frac{D_{rs}}{\Delta}|D^j_{1t}| + \frac{D_{st}}{\Delta}|D^j_{1r}| = 0 \qquad (III)$$

$$\frac{\partial D_{rs}}{\partial x_t} - \frac{\partial D_{st}}{\partial x_r} + \sum_j \left(D_{jt}\left\{ {rs \atop j} \right\} - D_{jr}\left\{ {st \atop j} \right\} \right) = 0 \qquad (IV)$$

$t, r, s = 1, 2, 3;\ t \neq r;\ t \neq s;\ j = 1, 2, 3.$

From (III) we derive the twenty four equations

$$\left.\begin{aligned}
&\{1j\,12\} - \frac{D_{11}}{\Delta}|D^j_{12}| + \frac{D_{12}}{\Delta}|D^j_{11}| = 0\\
&\{1j\,13\} - \frac{D_{11}}{\Delta}|D^j_{13}| + \frac{D_{13}}{\Delta}|D^j_{11}| = 0\\
&\{2j\,21\} - \frac{D_{22}}{\Delta}|D^j_{11}| + \frac{D_{12}}{\Delta}|D^j_{12}| = 0\\
&\{2j\,23\} - \frac{D_{22}}{\Delta}|D^j_{13}| + \frac{D_{23}}{\Delta}|D^j_{12}| = 0\\
&\{3j\,31\} - \frac{D_{33}}{\Delta}|D^j_{11}| + \frac{D_{13}}{\Delta}|D^j_{13}| = 0\\
&\{3j\,32\} - \frac{D_{33}}{\Delta}|D^j_{12}| + \frac{D_{23}}{\Delta}|D^j_{13}| = 0\\
&\{3j\,21\} - \frac{D_{23}}{\Delta}|D^j_{11}| + \frac{D_{13}}{\Delta}|D^j_{12}| = 0\\
&\{2j\,31\} - \frac{D_{23}}{\Delta}|D^j_{11}| + \frac{D_{12}}{\Delta}|D^j_{13}| = 0
\end{aligned}\right\} \qquad (III)$$

Which are a generalization of the four* ~~eq~~ equations of the the theory of surfaces giving the curvature of the first fundamental form.

* Bianchi — Lezioni p. 90.

From II I get the eight equations

$$\frac{\partial\lambda_{11}}{\partial u}-\frac{\partial\lambda_{12}}{\partial t}+\left\{{11 \atop 1}\right\}\lambda_{21}+\left\{{11 \atop 2}\right\}\lambda_{22}+\left\{{11 \atop 3}\right\}\lambda_{23}-\left\{{12 \atop 1}\right\}\lambda_{11}-\left\{{12 \atop 2}\right\}\lambda_{12}-\left\{{12 \atop 3}\right\}\lambda_{13}=0$$

$$\frac{\partial\lambda_{11}}{\partial v}-\frac{\partial\lambda_{13}}{\partial t}+\left\{{11 \atop 1}\right\}\lambda_{31}+\left\{{11 \atop 2}\right\}\lambda_{32}+\left\{{11 \atop 3}\right\}\lambda_{33}-\left\{{13 \atop 1}\right\}\lambda_{11}-\left\{{13 \atop 2}\right\}\lambda_{12}-\left\{{13 \atop 3}\right\}\lambda_{13}=0$$

$$\frac{\partial\lambda_{22}}{\partial t}-\frac{\partial\lambda_{12}}{\partial u}+\left\{{22 \atop 1}\right\}\lambda_{11}+\left\{{22 \atop 2}\right\}\lambda_{12}+\left\{{22 \atop 3}\right\}\lambda_{13}-\left\{{12 \atop 1}\right\}\lambda_{21}-\left\{{12 \atop 2}\right\}\lambda_{22}-\left\{{12 \atop 3}\right\}\lambda_{23}=0$$

$$\left.\begin{aligned}
&\frac{\partial\lambda_{22}}{\partial v}-\frac{\partial\lambda_{23}}{\partial u}+\left\{{22 \atop 1}\right\}\lambda_{31}+\left\{{22 \atop 2}\right\}\lambda_{32}+\left\{{22 \atop 3}\right\}\lambda_{33}-\left\{{23 \atop 1}\right\}\lambda_{21}-\left\{{23 \atop 2}\right\}\lambda_{22}-\left\{{23 \atop 3}\right\}\lambda_{23}=0\\
&\frac{\partial\lambda_{33}}{\partial t}-\frac{\partial\lambda_{13}}{\partial v}+\left\{{33 \atop 1}\right\}\lambda_{11}+\left\{{33 \atop 2}\right\}\lambda_{12}+\left\{{33 \atop 3}\right\}\lambda_{13}-\left\{{13 \atop 1}\right\}\lambda_{31}-\left\{{13 \atop 2}\right\}\lambda_{32}-\left\{{13 \atop 3}\right\}\lambda_{33}=0
\end{aligned}\right\}\ (\mathrm{IV})$$

$$\frac{\partial\lambda_{33}}{\partial u}-\frac{\partial\lambda_{23}}{\partial v}+\left\{{33 \atop 1}\right\}\lambda_{21}+\left\{{33 \atop 2}\right\}\lambda_{22}+\left\{{33 \atop 3}\right\}\lambda_{23}-\left\{{23 \atop 1}\right\}\lambda_{31}-\left\{{23 \atop 2}\right\}\lambda_{32}-\left\{{23 \atop 3}\right\}\lambda_{33}=0$$

$$\frac{\partial\lambda_{23}}{\partial t}-\frac{\partial\lambda_{13}}{\partial u}+\left\{{23 \atop 1}\right\}\lambda_{11}+\left\{{23 \atop 2}\right\}\lambda_{12}+\left\{{23 \atop 3}\right\}\lambda_{13}-\left\{{13 \atop 1}\right\}\lambda_{21}-\left\{{13 \atop 2}\right\}\lambda_{22}-\left\{{13 \atop 3}\right\}\lambda_{23}=0$$

$$\frac{\partial\lambda_{23}}{\partial t}-\frac{\partial\lambda_{12}}{\partial v}+\left\{{23 \atop 1}\right\}\lambda_{11}+\left\{{23 \atop 2}\right\}\lambda_{12}+\left\{{23 \atop 3}\right\}\lambda_{13}-\left\{{12 \atop 1}\right\}\lambda_{31}-\left\{{12 \atop 2}\right\}\lambda_{32}-\left\{{12 \atop 3}\right\}\lambda_{33}=0$$

which are a generalization of the Mainardi-Codazzi equations.

Hence, given the differential quadratic form

$$\varphi=E_{11}dt^2+E_{22}du^2+E_{33}dv^2+\text{------}$$

which is definite and whose coefficients satisfy (III) and (IV) there

exist a hypersurface admitting this as linear element.

Ricci* has deduced the necessary and sufficient conditions that a differential quadratic form in n variables be irreducible and of the first class, and it is interesting to note that these are exactly the conditions (III) and (IV) for the case of three variables.

Introduce the Christoffel symbols of four indices of the first kind by

$$(rkih) = \sum_\nu a_{\nu k}\{r\nu ih\}$$

These enjoy the following properties**

$$(rkih) = -(krih)$$

$$(rkih) = -(rkhi)$$

$$(rkih) = (ihrk)$$

$$(rkih) + (rihk) + (rhki) = 0$$

Taking account of these relations there are six symbols of the first kind

* Loc. Cit.

** Bianchi – Lezioni, p. 80.

not identically zero:

$(1212), (1213), (1223), (1313), (1323), (2323).$

Introduce the symbol of two indices

$$(lq)\sqrt{a} = \begin{vmatrix} \frac{\partial^2 y_1}{\partial x_l \partial x_q} & \frac{\partial^2 y_2}{\partial x_l \partial x_q} & \frac{\partial^2 y_3}{\partial x_l \partial x_q} & \frac{\partial^2 y_4}{\partial x_l \partial x_q} \\ \frac{\partial y_1}{\partial x_1} & - & - & - \\ \frac{\partial y_1}{\partial x_2} & - & - & - \\ \frac{\partial y_1}{\partial x_3} & - & - & - \end{vmatrix}$$

and the symbol of three indices

$$(lmp) = \frac{\partial(lp)}{\partial x_m} - \frac{\partial(lm)}{\partial x_p} + \sum_{rs} a_{rs}\left[(rm)\begin{bmatrix} lp \\ s \end{bmatrix} - (rp)\begin{bmatrix} lm \\ s \end{bmatrix}\right]$$

Then the necessary and sufficient conditions that the differential quadratic form

$$f = \sum_{rs}^{n} a_{rs}\, dx_r\, dx_s$$

be irreducible and of first class are

$a \neq 0$; $(lmpq) = (lp)(mq) - (lq)(mp)$, $(lmp) = 0$

To identify these with equations (III) and (IV) we have simply to write

$x = y_1,\ y = y_2,\ z = y_3,\ w = y_4,\ t = x_1,\ u = x_2,\ v = x_3$

and we see at once from (3*) p. 17 that $(ig) = D_{ig}$ coefficient of the second form since the symbols $(imp\,g)$ are the minors of the second order in the discriminant of the second form

$$D = \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}$$

Solving the system of equations

$$(rkih) = \Sigma a_{rk} \{rrih\}$$

we have

$$\{rrih\} = \Sigma_k A_{rk} (rkih)$$

and substituting

$$(rkih) = (ri)(kh) - (rh)(ki)$$

we have at once

$$\{rrih\} - \Sigma_k A_{rk} (ri)(kh) - \Sigma_k A_{rk} (rh)(ki) = 0$$

or

$$\{r\,r\,i\,h\} - \frac{\omega_{ri}}{\Delta}|\omega^r_{ih}| + \frac{\omega_{rh}}{\Delta}|\omega^r_{ih}| = 0 \qquad (c)$$

Substituting $\omega_{lp} = (lp)$ in

$$(lmp) = 0$$

we get

$$\frac{\partial \omega_{lp}}{\partial x_m} - \frac{\partial \omega_{lm}}{\partial x_p} + \sum_r \left(\omega_{rm}\left\{{lp \atop r}\right\} - \omega_{rp}\left\{{lm \atop r}\right\}\right) = 0 \qquad (d)$$

These equations (c) and (d) are exactly equations (III) and (IV) and we may say;

In order that a differential quadratic expression in two variables represent the linear element of a hypersurface it must be irreducible and of the first class.

Equations (III) and (IV) can be written in a form which will be useful when the existence of a hypersurface corresponding to the two fundamental forms is considered.

From (II) we have at once

$$\frac{1}{\sqrt{a}}\frac{\partial \mathfrak{d}_{lp}}{\partial x_m} - \frac{1}{\sqrt{a}}\frac{\partial \mathfrak{d}_{lm}}{\partial x_p} + \frac{1}{\sqrt{a}}\left[\mathfrak{d}_{lm}\left\{{lp \atop l}\right\} - \mathfrak{d}_{lp}\left\{{lm \atop l}\right\}\right] + \frac{1}{\sqrt{a}}{\sum_r}'\left[\mathfrak{d}_{rm}\left\{{lp \atop r}\right\} - \mathfrak{d}_{rp}\left\{{lm \atop r}\right\}\right] = 0 \qquad (a)$$

where in Σ' $r = l$ is excluded.

Now $\frac{\partial \log\sqrt{a}}{\partial x_l} = \sum_{ik} a_{ik}\left[{il \atop k}\right]$

whence

$$\frac{\partial\left(\frac{1}{\sqrt{a}}\right)}{\partial x_l} = \sum_i \frac{1}{\sqrt{a}}\left\{{il \atop i}\right\}$$

and we have

$$\mathfrak{d}_{lp}\frac{\partial\left(\frac{1}{\sqrt{a}}\right)}{\partial x_m} - \mathfrak{d}_{lm}\frac{\partial\left(\frac{1}{\sqrt{a}}\right)}{\partial x_p} + \frac{1}{\sqrt{a}}\left(\mathfrak{d}_{lp}\left\{{lm \atop l}\right\} - \mathfrak{d}_{lm}\left\{{lp \atop l}\right\}\right) + \frac{1}{\sqrt{a}}{\sum_r}'\left(\mathfrak{d}_{lp}\left\{{rm \atop r}\right\} - \mathfrak{d}_{lm}\left\{{rp \atop r}\right\}\right) = 0 \qquad (b)$$

Adding (a) and (b)

$$\frac{\partial\left(\frac{\mathfrak{d}_{lp}}{\sqrt{a}}\right)}{\partial x_m} - \frac{\partial\left(\frac{\mathfrak{d}_{lm}}{\sqrt{a}}\right)}{\partial x_p} + \frac{1}{\sqrt{a}}\sum_r\left(\mathfrak{d}_{rm}\left\{{lp \atop r}\right\} - \mathfrak{d}_{rp}\left\{{lm \atop r}\right\} + \mathfrak{d}_{lp}\left\{{rm \atop r}\right\} - \mathfrak{d}_{lm}\left\{{rp \atop r}\right\}\right) = 0 \qquad (\mathrm{I}^*)$$

These hold whatever be the coordinate lines. Suppose the hypersurface referred to a triply orthogonal system, while at the same time reduces the standard form to a sum of squares

It will appear later that this is the system formed by the lines of curvature. In this case three only of the independent symbols (the pq) are different from zero, namely

$$(i_j i_j) = \partial_{ii}\,\partial_{jj} \quad (i, j = 1, 2, 3;\ i \neq j).$$

The remaining three not identically zero are

$$(i_j j k) = 0 \quad (i, j, k = 1, 2, 3, \ i \neq j \neq k).$$

From

$$\{rs\,ik\} = \sum_h \varepsilon_{rh}(rs\,ih)$$

$$= \varepsilon_{rr}(rs\,ik)$$

it is seen at once

$$\{ij\,ij\} = \frac{\partial_{ii}\,\partial_{jj}}{\varepsilon_{jj}}, \quad \{ji\,ji\} = \frac{\partial_{ii}\,\partial_{jj}}{\varepsilon_{ii}}$$

$$\{ij\,jk\} = 0 \quad (i \neq j \neq k)$$

Equations (III) then become

$$\{ij\,ij\} = \frac{\partial_{ii}\,\partial_{jj}}{\varepsilon_{jj}}, \quad \{ij\,jk\} = 0 \quad i \neq j \neq k \qquad (\mathrm{III}^*)$$

and equations (II) take the form

$$\frac{\partial\left(\frac{\lambda_{ii}}{\sqrt{a}}\right)}{\partial x_u} + \frac{\lambda_{uu}}{\sqrt{a}}\left\{ {ii \atop u} \right\} = 0 \qquad (II^{*})$$

Calculating these we have

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t}\right) + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u}\right) + \frac{1}{E_{33}}\frac{\partial\sqrt{E_{11}}}{\partial v}\frac{\partial\sqrt{E_{22}}}{\partial v} + \frac{\lambda_{11}\lambda_{22}}{\sqrt{E_{11}E_{22}}} = 0\\
&\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}\right) + \frac{1}{E_{22}}\frac{\partial\sqrt{E_{11}}}{\partial u}\frac{\partial\sqrt{E_{33}}}{\partial u} + \frac{\lambda_{11}\lambda_{33}}{\sqrt{E_{11}E_{33}}} = 0\\
&\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}\right) + \frac{1}{E_{11}}\frac{\partial\sqrt{E_{22}}}{\partial t}\frac{\partial\sqrt{E_{33}}}{\partial t} + \frac{\lambda_{22}\lambda_{33}}{\sqrt{E_{22}E_{33}}} = 0\\
&\frac{\partial^2\sqrt{E_{33}}}{\partial t\,\partial u} - \frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{11}}}{\partial u}\frac{\partial\sqrt{E_{33}}}{\partial t} - \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial t}\frac{\partial\sqrt{E_{33}}}{\partial u} = 0\\
&\frac{\partial^2\sqrt{E_{22}}}{\partial t\,\partial v} - \frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{11}}}{\partial v}\frac{\partial\sqrt{E_{22}}}{\partial t} - \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}\frac{\partial\sqrt{E_{33}}}{\partial t} = 0\\
&\frac{\partial^2\sqrt{E_{11}}}{\partial u\,\partial v} - \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u}\frac{\partial\sqrt{E_{22}}}{\partial v} - \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}\frac{\partial\sqrt{E_{33}}}{\partial u} = 0
\end{aligned}\right\} (III^{**})$$

The last three of these may also be written

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t}\right) - \frac{1}{\sqrt{E_{11}E_{22}}}\frac{\partial\sqrt{E_{22}}}{\partial t}\cdot\frac{\partial\sqrt{E_{33}}}{\partial u} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}\right) - \frac{1}{\sqrt{E_{11}E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}\frac{\partial\sqrt{E_{22}}}{\partial t} = 0$$

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}\right) - \frac{1}{\sqrt{E_{22}E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial u}\frac{\partial\sqrt{E_{22}}}{\partial v} = 0$$

From (IV*) we have

$$
\left.\begin{aligned}
&\frac{\partial}{\partial u}\left(\frac{D_{11}}{\sqrt{E_{11}}}\right) - \frac{D_{22}}{E_{22}}\frac{\partial\sqrt{E_{11}}}{\partial u} = 0\\
&\frac{\partial}{\partial v}\left(\frac{D_{11}}{\sqrt{E_{11}}}\right) - \frac{D_{33}}{E_{33}}\frac{\partial\sqrt{E_{11}}}{\partial v} = 0\\
&\frac{\partial}{\partial t}\left(\frac{D_{22}}{\sqrt{E_{22}}}\right) - \frac{D_{11}}{E_{11}}\frac{\partial\sqrt{E_{22}}}{\partial t} = 0\\
&\frac{\partial}{\partial v}\left(\frac{D_{22}}{\sqrt{E_{22}}}\right) - \frac{D_{33}}{E_{33}}\frac{\partial\sqrt{E_{22}}}{\partial v} = 0\\
&\frac{\partial}{\partial t}\left(\frac{D_{33}}{\sqrt{E_{33}}}\right) - \frac{D_{11}}{E_{11}}\frac{\partial\sqrt{E_{33}}}{\partial t} = 0\\
&\frac{\partial}{\partial u}\left(\frac{D_{33}}{\sqrt{E_{33}}}\right) - \frac{D_{22}}{E_{22}}\frac{\partial\sqrt{E_{33}}}{\partial u} = 0
\end{aligned}\right\} \qquad (I^{**})
$$

The relations (III) and (IV) which connect the coefficients of the two forms give the necessary and sufficient conditions which these must satisfy and we may say

Given the two differential quadratic forms

$$F = E_{11}\,dt^2 + E_{22}\,du^2 + E_{33}\,dv^2 + 2E_{12}\,dt\,du + 2E_{13}\,dt\,dv + 2E_{23}\,du\,dv$$

$$\Phi = D_{11}\,dt^2 + D_{22}\,du^2 + D_{33}\,dv^2 + 2D_{12}\,dt\,du + 2D_{13}\,dt\,dv + 2D_{23}\,du\,dv$$

of which the first is definite, then in order that there exist a hypersurface admitting these as first and second fundamental forms it is necessary and sufficient that relations (III) and (II) be satisfied. The corresponding hypersurface is determined to within translation* in hyperspace.

Suppose the hypersurface whose existence and uniqueness we wish to demonstrate under the hypotheses above to be referred to its lines of curvature. Then (as will be shown later)

$$E_{12} = E_{13} = E_{23} = D_{12} = D_{13} = D_{23} = 0$$

and except in the case where $E_i = \lambda D_i$ these lines of curvature are uniquely determined. Consider at every point of the hypersurface the tri-rectangular tetrahedroid formed by the tangents to

* Jordan – Essai sur la géométrie à n dimensions. Cinématique [illegible]

the positive directions of these lines and the normal to the hypersurface. Let (X_1, Y_1, Z_1, W_1), (X_2, Y_2, Z_2, W_2), (X_3, Y_3, Z_3, W_3), (X_4, Y_4, Z_4, W_4) be the respective direction cosines of these lines.

We then have

$$X_1 = \frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}, \quad Y_1 = \frac{1}{\sqrt{E_{11}}}\frac{\partial y}{\partial t}, \quad Z_1 = \frac{1}{\sqrt{E_{11}}}\frac{\partial z}{\partial t}, \quad W_1 = \frac{1}{\sqrt{E_{11}}}\frac{\partial w}{\partial t}$$

$$X_2 = \frac{1}{\sqrt{E_{22}}}\frac{\partial x}{\partial u}, \quad - \; - \; - \; - \; - \; - \; - \; -$$

$$X_3 = \frac{1}{\sqrt{E_{33}}}\frac{\partial x}{\partial v}, \quad - \; - \; - \; - \; - \; - \; -$$

$$X_4 = X, \quad - \; - \; - \; - \; - \; - \; - \; -$$

From formulae (I) and (II), page 19, substituting for the Christoffel symbols their values from page 15, we have

$$\frac{\partial X_1}{\partial t} = -\frac{X_2}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u} - \frac{X_3}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v} + \frac{d_{11}}{\sqrt{E_{11}}}X_4$$

$$\frac{\partial X_1}{\partial u} = \frac{X_2}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t}, \quad \frac{\partial X_1}{\partial v} = \frac{X_3}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t}$$

$$\frac{\partial X_2}{\partial t} = \frac{X_1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u}, \quad \frac{\partial X_2}{\partial v} = \frac{X_3}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u}$$

$$\frac{\partial X_2}{\partial u} = -\frac{X_1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t} - \frac{X_3}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v} + \frac{d_{22}}{\sqrt{E_{22}}}X_4$$

$$\frac{\partial X_3}{\partial t} = \frac{X_1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}, \quad \frac{\partial X_3}{\partial u} = \frac{X_2}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}$$

$$\frac{\partial X_3}{\partial v} = -\frac{X_1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t} - \frac{X_2}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u} + \frac{\partial_{33}}{\sqrt{E_{33}}} X_4$$

$$\frac{\partial X_4}{\partial t} = -\frac{\partial_{11}}{\sqrt{E_{11}}} X_1, \quad \frac{\partial X_4}{\partial u} = -\frac{\partial_{22}}{\sqrt{E_{22}}} X_2, \quad \frac{\partial X_4}{\partial v} = -\frac{\partial_{33}}{\sqrt{E_{33}}} X_3$$

The unknown functions $(X_1 X_2 X_3 X_4)$ $(Y_1 Y_2 Y_3 Y_4)$, $(Z_1 Z_2 Z_3 Z_4)$, $(W_1 W_2 W_3 W_4)$ then satisfy the four simultaneous linear homogeneous total differential equations

$$\left.\begin{aligned}
d\Theta_1 &= \left\{-\frac{\Theta_2}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u} - \frac{\Theta_3}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v} + \frac{\partial_{11}}{\sqrt{E_{11}}}\Theta_4\right\}dt + \frac{\Theta_2}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t}du + \frac{\Theta_3}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t}dv \\
d\Theta_2 &= \frac{\Theta_1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u}dt + \left\{-\frac{\Theta_1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t} - \frac{\Theta_3}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v} + \frac{\partial_{22}}{\sqrt{E_{22}}}\Theta_4\right\}du + \frac{\Theta_3}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u}dv \\
d\Theta_3 &= \frac{\Theta_1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}dt + \frac{\Theta_2}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}du + \left\{-\frac{\Theta_1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t} - \frac{\Theta_2}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u} + \frac{\partial_{33}}{\sqrt{E_{33}}}\Theta_4\right\}dv \\
d\Theta_4 &= -\frac{\partial_{11}}{\sqrt{E_{11}}}\Theta_1 dt - \frac{\partial_{22}}{\sqrt{E_{22}}}\Theta_2 du - \frac{\partial_{33}}{\sqrt{E_{33}}}\Theta_3 dv
\end{aligned}\right\} \quad (4)$$

This is an illimitably integrable system or consider the condition of integrability

$$\frac{\partial}{\partial u}\left(\frac{\partial\Theta_1}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial\Theta_1}{\partial u}\right)$$

Developing this it reduces to

$$\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{11}}}{\partial u}\right)+\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t}\right)+\frac{1}{E_{33}}\frac{\partial\sqrt{E_{11}}}{\partial v}\frac{\partial\sqrt{E_{22}}}{\partial v}+\frac{\partial_{11}\partial_{22}}{\sqrt{E_{11}E_{33}}}\right]\Theta_2$$

$$+\left[-\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial v}\right)+\frac{1}{\sqrt{E_{22}E_{33}}}\frac{\partial\sqrt{E_{11}}}{\partial u}\frac{\partial\sqrt{E_{22}}}{\partial v}\right]\Theta_3$$

$$+\left[\frac{\partial}{\partial u}\left(\frac{\partial_{11}}{\sqrt{E_{11}}}\right)-\frac{\partial_{22}}{E_{22}}\frac{\partial\sqrt{E_{11}}}{\partial u}\right]\Theta_4=0$$

which is satisfied in virtue of equations (III**) and (IV**)

Hence there exists an integral system and a single one, which for initial values of the variables

$$t=t_0,\quad u=u_0,\quad v=v_0$$

reduces to arbitrarily given initial values. If $(\Theta_1\,\Theta_2\,\Theta_3\,\Theta_4)$, $(\Theta_1'\,\Theta_2'\,\Theta_3'\,\Theta_4')$ are two integral systems, then

$$\Theta_1\Theta_1'+\Theta_2\Theta_2'+\Theta_3\Theta_3'+\Theta_4\Theta_4'=\text{const.}$$

since the total differential of the first member is identically zero in virtue of equations (4).

Let $(X_1\, X_2\, X_3\, X_4)$ $(Y_1\, Y_2\, Y_3\, Y_4)$ $(Z_1\, Z_2\, Z_3\, Z_4)$ $(W_1\, W_2\, W_3\, W_4)$ be four integral systems of (4) which for $t = t_0$, $u = u_0$, $v = v_0$ reduce to the sixteen coefficients of an orthogonal substitution. Then it follows from the observation above that for all values of the variables will these sixteen quantities be the coefficients of an orthogonal substitution, and in particular

$$X_i^2 + Y_i^2 + Z_i^2 + W_i^2 = 1 \qquad (i = 1, 2, 3, 4)$$

$$X_i X_j + Y_i Y_j + Z_i Z_j + W_i W_j = 0 \qquad (i, j = 1, 2, 3, 4,\ i \neq j)$$

By (4) it is easily seen that the four expressions

$$\sqrt{E_{11}}\,\Theta_1\,dt + \sqrt{E_{22}}\,\Theta_2\,du + \sqrt{E_{33}}\,\Theta_3\,dv, \qquad \Theta = X, Y, Z, W.$$

are exact differentials and writing

$$\sigma = \int \sqrt{E_{11}}\,\Theta_1\,dt + \sqrt{E_{22}}\,\Theta_2\,du + \sqrt{E_{33}}\,\Theta_3\,dv$$

$$\sigma = x, y, z, w; \qquad \Theta_i = X_i, Y_i, Z_i, W_i.$$

we have a hypersurface with the given fundamental forms.

Consider the system of equations (4). This is identical with the systems (34)(34') (34'') found by Professor Craig,* when $\alpha, \beta, \gamma, \delta$ are replaced by $\Theta_1, \Theta_2, \Theta_3, \Theta_4$, and the a_{ij} by the coefficients in (4). Now Professor Craig has shown that the integration of system (34) can be reduced to the integration of a generalized** Riccati equation and in the same way the integration of system (4) can be reduced to the integration of the three simultaneous generalized Riccati equations below by the substitutions.

$$\Theta_1 = \frac{2\lambda}{\kappa^2+1}, \quad \Theta_2 = \frac{2\mu}{\kappa^2+1}, \quad \Theta_3 = \frac{2\nu}{\kappa^2+1}, \quad \Theta_4 = \frac{\kappa^2-1}{\kappa^2+1}$$

$$\kappa^2 = \lambda^2 + \mu^2 + \nu^2$$

$$\left.\begin{aligned}
\frac{\partial \lambda}{\partial t} &= -\frac{1}{\sqrt{E_{22}}}\frac{\partial \sqrt{E_{11}}}{\partial u}\,\mu - \frac{1}{\sqrt{E_{33}}}\frac{\partial \sqrt{E_{11}}}{\partial v}\,\nu + \frac{\kappa^2-1}{2}\frac{D_{11}}{\sqrt{E_{11}}} - \lambda^2 \frac{D_{11}}{\sqrt{E_{11}}} \\
\frac{\partial \mu}{\partial t} &= \frac{1}{\sqrt{E_{22}}}\frac{\partial \sqrt{E_{11}}}{\partial u}\,\lambda - \lambda\mu \frac{D_{11}}{\sqrt{E_{11}}} \\
\frac{\partial \nu}{\partial t} &= \frac{1}{\sqrt{E_{33}}}\frac{\partial \sqrt{E_{11}}}{\partial v}\,\lambda - \lambda\nu \frac{D_{11}}{\sqrt{E_{11}}}
\end{aligned}\right\}$$

* American Journal of Math. Vol. XX No. 2, p. 45.

** Loc. Cit., p. 41.

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial u} &= \frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t}\mu - \mu\lambda\frac{D_{22}}{\sqrt{E_{22}}}\\
\frac{\partial\mu}{\partial u} &= -\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{22}}}{\partial t}\lambda - \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}\nu + \frac{(\kappa^2-1)}{2}\frac{D_{22}}{\sqrt{E_{22}}} - \mu^2\frac{D_{22}}{\sqrt{E_{22}}}\\
\frac{\partial\nu}{\partial u} &= \frac{1}{\sqrt{E_{33}}}\frac{\partial\sqrt{E_{22}}}{\partial v}\mu - \mu\nu\frac{D_{22}}{\sqrt{E_{22}}}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial v} &= \frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t}\nu - \nu\lambda\frac{D_{33}}{\sqrt{E_{33}}}\\
\frac{\partial\mu}{\partial v} &= \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u}\nu - \nu\mu\frac{D_{33}}{\sqrt{E_{33}}}\\
\frac{\partial\nu}{\partial v} &= -\frac{1}{\sqrt{E_{11}}}\frac{\partial\sqrt{E_{33}}}{\partial t}\lambda - \frac{1}{\sqrt{E_{22}}}\frac{\partial\sqrt{E_{33}}}{\partial u}\mu + \frac{(\kappa^2-1)}{2}\frac{D_{33}}{\sqrt{E_{33}}} - \nu^2\frac{D_{33}}{\sqrt{E_{33}}}
\end{aligned}\right\}$$

Here we have chosen the special tetraedroid formed by the tangents of the lines of curvature and the normal but any other tetrarectangular tetrahedroid might have been taken and we should have arrived at a set of equations similar to (4) illimitably integrable in virtue of relations (III) and (II) and these would have led to three generalized Riccati equations of the form

$$\left.\begin{aligned}
\frac{\partial\lambda}{\partial t} &= a_1\mu + b_1\nu + \left(\frac{k^2}{2}-1\right)c_1 + \lambda(a_1'\lambda + b_1'\mu + c_1'\nu) \\
\frac{\partial\mu}{\partial t} &= a_2\lambda + b_2\nu + \left(\frac{k^2}{2}-1\right)c_2 + \mu(a_2'\lambda + b_2'\mu + c_2'\nu) \\
\frac{\partial\nu}{\partial t} &= a_3\lambda + b_3\mu + \left(\frac{k^2}{2}-1\right)c_3 + \nu(a_3'\lambda + b_3'\mu + c_3'\nu)
\end{aligned}\right\}$$

We may then say:

Given the two fundamental forms, the first being definite, and the coefficients of which satisfy (III) and (II), there exists a unique hypersurface admitting these as first and second fundamental forms, and to effectively obtain this hypersurface it is necessary to integrate three simultaneous generalized Riccati equations.

The integration of a single generalized Riccati when particular solutions are known has been studied by Mr. John Eiesland.*

* Note on the Integration of a Certain System of Differential Equations. American Journal of Math. Vol XX No. 3

III

Lines on the Hypersurface

In treating of hypersurfaces the definition of lines of curvature these are lines along which the normals of the hypersurface form a developable surface. Let M be a point on the hypersurface and M_1 the point where the normal touches the edge of regression of the developable. Along a line of curvature x, y, z, w and X, Y, Z, W are functions of a single parameter the arc s say. Then

$$x_1 = x - rX,\ y_1 = y - rY,\ z_1 = z - rZ,\ w_1 = w - rW$$

where r is the algebraic value of MM_1.

Differentiating with respect to the arc

$$\lambda X = \frac{dx}{ds} - r\frac{dX}{ds} - X\frac{dr}{ds}$$

$$\lambda Y = \frac{dy}{ds} - r\frac{dY}{ds} - Y\frac{dr}{ds}$$

$$\lambda Z = \frac{dz}{ds} - r\frac{dZ}{ds} - Z\frac{dr}{ds}$$

$$\lambda W = \frac{dw}{ds} - r\frac{dW}{ds} - W\frac{dr}{ds}$$

Multiplying by X, Y, Z, W in order and summing we have $\rho = -\frac{dr}{ds}$ and hence

$$\frac{dx}{ds} = r\frac{dX}{ds}, \quad \frac{dy}{ds} = r\frac{dY}{ds}$$

$$\frac{dz}{ds} = r\frac{dZ}{ds} \quad \frac{dw}{ds} = r\frac{dW}{ds}$$

Hence along a line of curvature

$$\frac{dx}{dX} = \frac{dy}{dY} = \frac{dz}{dZ} = \frac{dw}{dW} = r$$

Writing these four equations in curvilinear coordinates

$$\frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv = r\left(\frac{\partial X}{\partial t}dt + \frac{\partial X}{\partial u}du + \frac{\partial X}{\partial v}dv\right)$$

$$\frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv = r\left(\frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial u}du + \frac{\partial Y}{\partial v}dv\right)$$

$$\frac{\partial z}{\partial t}dt + \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv = r\left(\frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial u}du + \frac{\partial Z}{\partial v}dv\right)$$

$$\frac{\partial w}{\partial t}dt + \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv = r\left(\frac{\partial W}{\partial t}dt + \frac{\partial W}{\partial u}du + \frac{\partial W}{\partial v}dv\right)$$

Multiplying by $\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}, \frac{\partial w}{\partial t}$; $\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}, \frac{\partial w}{\partial u}$; $\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}, \frac{\partial w}{\partial v}$ in order and summing each time we have the equivalent system

$$\left.\begin{array}{l} \mathcal{E}_{11}\,dt+\mathcal{E}_{12}\,du+\mathcal{E}_{13}\,dv=-r(\partial_{11}dt+\partial_{12}du+\partial_{13}dv) \\ \mathcal{E}_{21}\,dt+\mathcal{E}_{22}\,du+\mathcal{E}_{23}\,dv=-r(\partial_{21}dt+\partial_{22}du+\partial_{23}dv) \\ \mathcal{E}_{31}\,dt+\mathcal{E}_{32}\,du+\mathcal{E}_{33}\,dv=-r(\partial_{31}dt+\partial_{32}du+\partial_{33}dv) \end{array}\right\} \quad (1)$$

These are the equations of the lines of curvature on the hypersurface. Ricci* was arrived at these same equations for the case of n variables by analogous reasoning and Voss** was deduced similar ones, also for n variables, from considerations on the geodesic lines in a curved space. This last method shows at once that the lines of curvature are orthogonal under any say.

Through every point on the hypersurface pass three lines of curvature forming a triply orthogonal system.

The three radii of curvature at the point are given by the roots of the cubic.

(* Loc. Cit., p. 64.

** Loc. Cit. p. 40

$$\begin{vmatrix} \varepsilon_{11}+r\vartheta_{11} & \varepsilon_{12}+r\vartheta_{12} & \varepsilon_{13}+r\vartheta_{13} \\ \varepsilon_{21}+r\vartheta_{21} & \varepsilon_{22}+r\vartheta_{22} & \varepsilon_{23}+r\vartheta_{23} \\ \varepsilon_{31}+r\vartheta_{31} & \varepsilon_{32}+r\vartheta_{32} & \varepsilon_{33}+r\vartheta_{33} \end{vmatrix} = 0 \qquad (2)$$

The product of the reciprocals of the three radii of curvature

$$\frac{1}{r_1 r_2 r_3} = -\frac{\vartheta}{A}$$

may be looked upon as the generalization of the total curvature of Gauss, and, as will be shown in the section on spherical representation, the ratio of the element of hyperarea on the sphere to the element of hyperarea on the hypersurface is equal to the total curvature of the hypersurface.

Developing equation (2) above we have

$$\frac{1}{r^3} - \frac{H}{r^2} + \frac{L}{r} - K = 0$$

Padova* has given a rather elegant form to the coefficients in this equation

* Sulle espressioni invariabile. Memorie della R. Acc. dei Lincei Serie 4 t. V.

Let $U(y_1 y_2 y_3 y_4) = 0$ be the equation of the hypersurface, and write

$$\Delta_1 U = \sum_{h}^{4} \left(\frac{\partial U}{\partial y_h}\right)^2$$

$$\Delta_{21} U = \sum_{1}^{4}{}_{h} \frac{\partial^2 U}{\partial y_h^2}$$

$$\Delta_{22} U = \sum_{1}^{4}{}_{hk} \left[\frac{\partial^2 U}{\partial y_h^2}\frac{\partial^2 U}{\partial y_k^2} - \left(\frac{\partial^2 U}{\partial y_h \partial y_k}\right)^2\right]$$

Padova then deduces the following values for H and L.

$$L = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{\Delta_{21} U}{\sqrt{\Delta_1 U}} - \frac{d\sqrt{\Delta_1 U}}{dU}$$

$$H = \frac{1}{r_1 r_2} + \frac{1}{r_1 r_3} + \frac{1}{r_2 r_3} = \frac{\Delta_{22} U}{\Delta_1 U} - \frac{\Delta_{21} U}{\sqrt{\Delta_1 U}} \cdot \frac{d\sqrt{\Delta_1 U}}{dU} + \frac{\Delta_1 \Delta_1 U}{4(\Delta_1 U)^2}$$

The particular case of this is of course that of a surface immersed in three dimensional Euclidean space and Lamé* had already given similar expressions for this. In a second

* Leçons sur les Coordonnées Curvilignes

Signor Fubini* extends these to a surface immersed in a hypersurface.

Writing $U(x_1 x_2 x_3) = \text{const}$ as the equation of the surface immersed in the hypersurface whose linear element is given by

$$ds^2 = \sum_1^3 rs\, a_{rs}\, dx_r\, dx_s$$

we finds

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{\Delta_2 U}{\sqrt{\Delta_1 U}} - \frac{d\sqrt{\Delta_1 U}}{dU}$$

$$\frac{1}{\rho_1 \rho_2} = \frac{\Delta_{22} U}{\Delta_1 U} - \frac{\Delta_2 U}{\sqrt{\Delta_1 U}} \frac{d\sqrt{\Delta_1 U}}{dU} + \frac{\Delta \Delta_1 U}{4(\Delta_1 U)^2}$$

where $\Delta_1 U$, $\Delta_2 U$, $\Delta_{22} U$ are differential parameters constructed with respect to the linear element of the hypersurface and ρ_1, ρ_2 satisfy

$$\begin{vmatrix} b_{11} - \rho \frac{\Omega_{11}}{\sqrt{\Delta_1 U}} & b_{12} - \rho \frac{\Omega_{12}}{\sqrt{\Delta_1 U}} \\ b_{12} - \rho \frac{\Omega_{12}}{\sqrt{\Delta_1 U}} & b_{22} - \rho \frac{\Omega_{22}}{\sqrt{\Delta_1 U}} \end{vmatrix} = 0$$

in which $b_{rs} = \sum_{ij} a_{ij} \frac{\partial x_i}{\partial q_r}\frac{\partial x_j}{\partial q_s}$

$$\Omega_{rs} = \sum_{hk} \frac{\partial^2 U}{\partial x_h \partial x_k}\frac{\partial x_h}{\partial q_r}\frac{\partial x_k}{\partial q_s} - \sum_{hkij} \begin{bmatrix} ij \\ k \end{bmatrix} A_{hk} \frac{\partial U}{\partial x_h}\frac{\partial x_i}{\partial q_r}\frac{\partial x_j}{\partial q_s}$$

where q_1 and q_2 are curvilinear coordinates on

$$U = \text{const.}$$

We see then that the partial differential equations which define the minima and developable surfaces immersed in a hypersurface have the same form when expressed in the differential parameters as those for surfaces in Euclidean space of three dimensions.

If we suppose the hypersurface referred to a triply orthogonal system x_1, x_2, x_3, then the condition that the parametric surface

$$x_1 = \text{const}$$

is a minimum surface is

$$\frac{\partial \log a_{22} a_{33}}{\partial x_1} = 0$$

the linear element of the hypersurface being

$$ds^2 = a_{11} dx_1^2 + a_{22} dx_2^2 + a_{33} dx_3^2$$

Let C be a curve on the hypersurface and consider t, u, v, x, y, z, w as functions of the arc. The direction cosines of the tangent to C at any point are given by

$$\cos\alpha = \frac{dx}{ds} = \frac{\partial x}{\partial t}\frac{dt}{ds} + \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds}$$

$$\cos\beta = \frac{dy}{ds} = \frac{\partial y}{\partial t}\frac{dt}{ds} + \frac{\partial y}{\partial u}\frac{du}{ds} + \frac{\partial y}{\partial v}\frac{dv}{ds}$$

$$\cos\gamma = \frac{dz}{ds} = \frac{\partial z}{\partial t}\frac{dt}{ds} + \frac{\partial z}{\partial u}\frac{du}{ds} + \frac{\partial z}{\partial v}\frac{dv}{ds}$$

$$\cos\delta = \frac{dw}{ds} = \frac{\partial w}{\partial t}\frac{dt}{ds} + \frac{\partial w}{\partial u}\frac{du}{ds} + \frac{\partial w}{\partial v}\frac{dv}{ds}$$

Let σ $(0 < \sigma < \pi)$ be the angle between the principal normal to C and the normal to the hypersurface. The direction cosines of the principal normal to C are*

* (Bianchi – Loc. Cit. p. 22)

$$= \frac{d\cos\alpha}{ds}, \; = \frac{d\cos\beta}{ds}, \; = \frac{d\cos\gamma}{ds}, \; = \frac{d\cos\delta}{ds}$$

where ρ is radius of first curvature of C.
We then have

$$\cos\sigma = \rho \sum X \frac{d\cos\alpha}{ds}$$

Hence we have at once by differentiating as indicated above

$$\frac{\cos\sigma}{\rho} = \frac{\mathfrak{D}_{11}dt^2 + \mathfrak{D}_{22}du^2 + \mathfrak{D}_{33}dv^2 + 2\mathfrak{D}_{12}dt\,du + 2\mathfrak{D}_{13}dt\,dv + 2\mathfrak{D}_{23}du\,dv}{\mathcal{E}_{11}dt^2 + \mathcal{E}_{22}du^2 + \mathcal{E}_{33}dv^2 + 2\mathcal{E}_{12}dt\,du + 2\mathcal{E}_{13}dt\,dv + 2\mathcal{E}_{23}du\,dv}$$

$$= \frac{\varphi}{ds^2}$$

Through the normal of the hypersurface and the tangent of C pass a plane making a normal section. The first curvature of this section is given by the formula above, making $\cos\sigma = \pm 1$. Let R denote the radius of first curvature of this normal section, we then have

$$\frac{1}{R} = \pm \frac{\cos\sigma}{\rho}$$

Hence $\quad \rho = \pm R \cos \sigma$

Meusnier's theorem then applies to the hypersurface, and therefore

"The radius of first curvature of a curve C traced on a hypersurface equals in every point M the radius of first curvature of the normal section made by the plane through the tangent to C at M multiplied by the cosine of the angle which this plane makes with the osculating plane of the curve C"

If R be taken positive when the direction from the centre of curvature of the normal section toward the point M is along the positive normal to the hypersurface, then

$$\frac{1}{R} = -\frac{\mathcal{E}}{ds^2}$$

Refer the hypersurface to the lines of curvature and denote the principal radii of curvature by r_1, r_2, r_3. Then along the t, u, v lines respectively

$dx = r_1\, dX$, $dy = r_1\, dY$, $dz = r_1\, dZ$, $dw = r_1\, dW$

$dx = r_2\, dX$, - - - - - - - - - - - -

$dx = r_3\, dX$, - - - - - - - - - - - -

or

$$\frac{\partial x}{\partial t} = r_1\frac{\partial X}{\partial t},\ \frac{\partial y}{\partial t} = r_1\frac{\partial Y}{\partial t},\ \frac{\partial z}{\partial t} = r_1\frac{\partial Z}{\partial t},\ \frac{\partial w}{\partial t} = r_1\frac{\partial W}{\partial t}$$

$$\frac{\partial x}{\partial u} = r_2\frac{\partial X}{\partial u}, \text{ - - - - - - - - - -}$$

$$\frac{\partial x}{\partial v} = r_3\frac{\partial X}{\partial v}, \text{ - - - - - - - - - -}$$

From these relations follows at once

$$\mathcal{E}_{11} = -r_1 \mathcal{D}_{11},\quad \mathcal{E}_{22} = -r_2 \mathcal{D}_{22},\quad \mathcal{E}_{33} = -r_3 \mathcal{D}_{33}$$

$$\mathcal{D}_{12} = \mathcal{D}_{13} = \mathcal{D}_{23} = 0$$

whence

$$\frac{1}{R} = \frac{\frac{\mathcal{E}_{11}}{r_1}dt^2 + \frac{\mathcal{E}_{22}}{r_2}du^2 + \frac{\mathcal{E}_{33}}{r_3}dv^2}{ds^2}$$

$$= \frac{\mathcal{E}_{11}}{r_1}\left(\frac{dt}{ds}\right)^2 + \frac{\mathcal{E}_{22}}{r_2}\left(\frac{du}{ds}\right)^2 + \frac{\mathcal{E}_{33}}{r_3}\left(\frac{dv}{ds}\right)^2$$

Denoting by α, β, γ the direction angles of the tangent to this normal section referred to the lines of curvature,

$$\frac{1}{R} = \frac{\cos^2\alpha}{r_1} + \frac{\cos^2\beta}{r_2} + \frac{\cos^2\gamma}{r_3}$$

Hence we know at once that r_1, r_2, r_3 are the radii of first curvature of the normal sections tangent to the lines of curvature.

On the tangent hyperplane fix a system of rectangular axes ξ, η, ζ, coinciding with the positive directions of the lines of curvature t, u, v and consider the conicoid

$$\frac{\xi^2}{r_1} + \frac{\eta^2}{r_2} + \frac{\zeta^2}{r_3} = 1$$

The length of any semi-diameter whose direction angles are α, β, γ is given by $\frac{1}{r^2} = \frac{\cos^2\alpha}{r_1} + \frac{\cos^2\beta}{r_2} + \frac{\cos^2\gamma}{r_3}$

Hence $c^2 = r$

and the square of every semidiameter of the indicatroid equals the radius of first curvature of the normal section whose plane passes through it.

Define as conjugate directions at a point on the hypersurface the directions of three conjugate diameters of the indicatroid at that point. Let $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$ denote the direction cosines of the conjugate tangents. Then

$$\frac{\alpha_1\alpha_2}{r_1} + \frac{\beta_1\beta_2}{r_2} + \frac{\gamma_1\gamma_2}{r_3} = 0$$

$$\frac{\alpha_2\alpha_3}{r_1} + \frac{\beta_2\beta_3}{r_2} + \frac{\gamma_2\gamma_3}{r_3} = 0$$

$$\frac{\alpha_3\alpha_1}{r_1} + \frac{\beta_3\beta_1}{r_2} + \frac{\gamma_3\gamma_1}{r_3} = 0$$

Still supposing the hypersurface referred to its lines of curvature denote

its displacements along the conjugate lines by $d\,\delta\,\delta'$. Then

$$\alpha_1 = \sqrt{E_{11}}\frac{dt}{ds},\quad \beta_1 = \sqrt{E_{22}}\frac{du}{ds},\quad \gamma_1 = \sqrt{E_{33}}\frac{dv}{ds}$$

$$\alpha_2 = \sqrt{E_{11}}\frac{\delta t}{\delta s},\quad \beta_2 = \sqrt{E_{22}}\frac{\delta u}{\delta s},\quad \gamma_2 = \sqrt{E_{33}}\frac{\delta v}{\delta s}$$

$$\alpha_3 = \sqrt{E_{11}}\frac{\delta' t}{\delta' s},\quad \beta_3 = \sqrt{E_{22}}\frac{\delta' u}{\delta' s},\quad \gamma_3 = \sqrt{E_{33}}\frac{\delta' v}{\delta' s}$$

and the conjugate lines are given by

$$\frac{E_{11}}{r_1}dt\,\delta t + \frac{E_{22}}{r_2}du\,\delta u + \frac{E_{33}}{r_3}dv\,\delta v = 0$$

$$\frac{E_{11}}{r_1}\delta t\,\delta' t + \frac{E_{22}}{r_2}\delta u\,\delta' u + \frac{E_{33}}{r_3}\delta v\,\delta' v = 0$$

$$\frac{E_{11}}{r_1}\delta' t\,dt + \frac{E_{22}}{r_2}\delta' u\,du + \frac{E_{33}}{r_3}\delta' v\,dv = 0$$

Now construct with respect to the second fundamental form, the equations analogous to the conditions of orthogonality for three lines constructed (with respect to the first form. These three equations will give the conjugate lines on the hypersurface expressed in any system

of coordinate lines. For we have

$$\partial_{11} dt\,\delta t + \partial_{22} du\,\delta u + \partial_{33} dv\,\delta v +$$
$$\partial_{12}(dt\,\delta u + \delta t\,du) + \partial_{13}(dt\,\delta v + \delta t\,dv) + \partial_{23}(du\,\delta v + \delta u\,dv) = 0$$
$$\partial_{11}\,\delta t\,\delta' t + \partial_{22}\,\delta u\,\delta' u + \partial_{33}\,\delta v\,\delta' v +$$
$$\partial_{12}(\delta t\,\delta' u + \delta u\,\delta' t) + \partial_{13}(\delta t\,\delta' v + \delta v\,\delta' t) + \partial_{23}(\delta u\,\delta' v + \delta v\,\delta' u) = 0$$
$$\partial_{11}\,\delta' t\,dt + \partial_{22}\,\delta' u\,du + \partial_{33}\,\delta' v\,dv +$$
$$\partial_{12}(\delta' t\,du + \delta' u\,dt) + \partial_{13}(\delta' t\,dv + \delta' v\,dt) + \partial_{23}(\delta' u\,dv + \delta' v\,du) = 0$$

and referring the hypersurface to its lines of curvature these reduce exactly to the three equations given above.

If the parametric lines form a conjugate system then these equations must be satisfied by

$$dt = dt \quad du = dv = 0$$
$$\delta t = 0,\ \delta u = \delta u,\ \delta v = 0$$
$$\delta' t = 0,\ \delta' u = 0 \quad \delta' v = \delta' v$$

whence $\quad \partial_{12} = \partial_{13} = \partial_{23} = 0$

and if these conditions are satisfied the equations above give the parametric lines.

Hence, the necessary and sufficient conditions that the coordinate lines form a conjugate system are

$$D_{12} = D_{13} = D_{23} = 0$$

Extending the definition of asymptotic lines on a surface a line is said to be asymptotic on a hypersurface if it coincides with its two conjugates. The asymptotic lines through a point then lie on the surface

$$D_{11}dt^2 + D_{22}du^2 + D_{33}dv^2 + 2D_{12}dt\,du + 2D_{13}dt\,dv + 2D_{23}du\,dv = 0$$

and at the point this coincides with the asymptotic cone of the indicatroid. In order that the parametric lines lie on this surface we must have

$$D_{11} = D_{22} = D_{33} = 0$$

Hence if the hypersurface is referred to any three of its asymptotic lines $D_{11} = D_{22} = D_{33} = 0$ and conversely.

If a plane moves so as to be constantly tangent to a developable surface it generates a developable hypersurface. The developable to which the moving plane is constantly tangent is termed the surface of regression and the edge of regression of the developable surface is termed the edge of regression of the hypersurface. The rectilinear generatrices of the surface of regression are there lines of curvature on the hypersurface and $K=0$. Conversely if $K=0$ we have a hypersurface with at least one line of curvature a right line. For referring to lines of curvature we have

$$\partial_{11}\,\partial_{22}\,\partial_{33}=0$$

and hence ∂_{11} say must vanish. From (II) p. 19 we have

$$\frac{\partial X}{\partial t} = \frac{\partial Y}{\partial t} = \frac{\partial Z}{\partial t} = \frac{\partial W}{\partial t} = 0$$

from which it follows that X, Y, Z, W, are functions of u and v only.
We also have

$$\frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}X + \frac{1}{\sqrt{E_{11}}}\frac{\partial y}{\partial t}Y + \frac{1}{\sqrt{E_{11}}}\frac{\partial z}{\partial t}Z + \frac{1}{\sqrt{E_{11}}}\frac{\partial w}{\partial t}W = 0$$

$$\frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}\frac{\partial X}{\partial u} + \text{---------} = 0$$

$$\frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}\frac{\partial X}{\partial v} + \text{---------} = 0$$

since the first expresses the orthogonality of the t line and the normal, while the second and third express the fact that $d_{12} = d_{13} = 0$

The direction cosines

$$\frac{1}{\sqrt{E_{11}}}\frac{\partial x}{\partial t}, \frac{1}{\sqrt{E_{11}}}\frac{\partial y}{\partial t}, \frac{1}{\sqrt{E_{11}}}\frac{\partial z}{\partial t}, \frac{1}{\sqrt{E_{1}}}\frac{\partial w}{\partial t}$$

of the t line are then functions of u and v only hence the t line is a right line and the hypersurface has one line of curvature a straight line.

The hypersurface of zero curvature is applicable to the hyperplane

For if $D = 0$ then the form

$$ds^2 = \mathcal{E}_{11}\,dt^2 + \mathcal{E}_{22}\,du^2 + \mathcal{E}_{33}\,dv^2 + 2\mathcal{E}_{12}\,dt\,du + 2\mathcal{E}_{13}\,dt\,dv + 2\mathcal{E}_{23}\,du\,dv$$

is of class zero*, and hence can be deduced from a differential quadratic form

$$ds^2 = dy_1^2 + dy_2^2 + dy_3^2.$$

IV

On the Differential Equations satisfied by the Coordinates of a Point on the Hypersurface.

From equations (I) page 19 we see that if $\vartheta_{12} = \vartheta_{13} = \vartheta_{23} = 0$ the coordinates of any point on the hypersurface satisfy the simultaneous partial differential equations of the form

$$\left.\begin{aligned}
\frac{\partial^2\theta}{\partial x_1\,\partial x_2} &= a_1\frac{\partial\theta}{\partial x_1} + b_1\frac{\partial\theta}{\partial x_2} + c_1\frac{\partial\theta}{\partial x_3}\\
\frac{\partial^2\theta}{\partial x_1\,\partial x_3} &= a_2\frac{\partial\theta}{\partial x_1} + b_2\frac{\partial\theta}{\partial x_2} + c_2\frac{\partial\theta}{\partial x_3}\\
\frac{\partial^2\theta}{\partial x_2\,\partial x_3} &= a_3\frac{\partial\theta}{\partial x_1} + b_3\frac{\partial\theta}{\partial x_2} - c_3\frac{\partial\theta}{\partial x_3}
\end{aligned}\right\}$$

* [illegible] — loc. cit., [illegible]

Conversely if the three simultaneous equations above admit four common solutions y_1, y_2, y_3, y_4, linearly independent, then the formulae

$y_1 = y_1(x_1 x_2 x_3),\ y_2 = y_2(x_1 x_2 x_3),\ y_3 = y_3(x_1 x_2 x_3),\ y_4 = y_4(x_1 x_2 x_3)$

define a hypersurface referred to its conjugate lines.

For

$$\begin{vmatrix} \frac{\partial^2 y_1}{\partial x_i \partial x_j} & \frac{\partial^2 y_2}{\partial x_i \partial x_j} & \frac{\partial^2 y_3}{\partial x_i \partial x_j} & \frac{\partial^2 y_4}{\partial x_i \partial x_j} \\ \frac{\partial y_1}{\partial x_1} & - & - & - \\ \frac{\partial y_1}{\partial x_2} & - & - & - \\ \frac{\partial y_1}{\partial x_3} & - & - & - \end{vmatrix} = 0 \quad \begin{matrix} (i, j = 1 2 3) \\ (i \neq j) \end{matrix}$$

and hence $D_{12} = D_{13} = D_{23} = 0$

If in addition to y_1, y_2, y_3, y_4 we have $\rho = y_1^2 + y_2^2 + y_3^2 + y_4^2$ as a common solution then

$$\frac{\partial^2 \rho}{\partial x_i \partial x_j} - \begin{Bmatrix} i j \\ 1 \end{Bmatrix} \frac{\partial \rho}{\partial x_1} - \begin{Bmatrix} i j \\ 2 \end{Bmatrix} \frac{\partial \rho}{\partial x_2} - \begin{Bmatrix} i j \\ 3 \end{Bmatrix} \frac{\partial \rho}{\partial x_3} = 2 \mathcal{E}_{ij} = 0$$

hence $\mathcal{E}_{12} = \mathcal{E}_{13} = \mathcal{E}_{23} = 0$ and the formulae above define a hypersurface referred

& its lines of curvature.

Again if $\partial_{11} = \partial_{22} = \partial_{33} = 0$ the coordinates satisfy three simultaneous equations of the type

$$\left.\begin{aligned}
\frac{\partial^2 \theta}{\partial x_1^2} &= a_1 \frac{\partial \theta}{\partial x_1} + b_1 \frac{\partial \theta}{\partial x_2} + c_1 \frac{\partial \theta}{\partial x_3} \\
\frac{\partial^2 \theta}{\partial x_2^2} &= a_2 \frac{\partial \theta}{\partial x_1} + b_2 \frac{\partial \theta}{\partial x_2} + c_2 \frac{\partial \theta}{\partial x_3} \\
\frac{\partial^2 \theta}{\partial x_3^2} &= a_3 \frac{\partial \theta}{\partial x_1} + b_3 \frac{\partial \theta}{\partial x_2} + c_3 \frac{\partial \theta}{\partial x_3}
\end{aligned}\right\} \qquad (B)$$

and conversely if these three equations admit four simultaneous solutions y_1, y_2, y_3, y_4 linearly independent then the formulae

$y_1 = y_1(x_1 x_2 x_3)$, $y_2 = y_2(x_1 x_2 x_3)$, $y_3 = y_3(x_1 x_2 x_3)$, $y_4 = y_4(x_1 x_2 x_3)$

define a hypersurface referred to three asymptotic lines.

For
$$\begin{vmatrix}
\frac{\partial^2 y_1}{\partial x_i^2} & \frac{\partial^2 y_2}{\partial x_i^2} & \frac{\partial^2 y_3}{\partial x_i^2} & \frac{\partial^2 y_4}{\partial x_i^2} \\
\frac{\partial y_1}{\partial x_1} & \cdots & \cdots & \cdots \\
\frac{\partial y_1}{\partial x_2} & \cdots & \cdots & \cdots \\
\frac{\partial y_1}{\partial x_3} & & \cdots &
\end{vmatrix} = 0 \qquad (i = 1, 2, 3)$$

and hence

$$D_{11} = D_{22} = D_{33} = 0$$

A projective transformation preserves the conjugate and asymptotic lines for a projective transformation is defined by

$$y_1 = \frac{\alpha_1}{\epsilon},\ y_2 = \frac{\alpha_2}{\epsilon},\ y_3 = \frac{\alpha_3}{\epsilon},\ y_4 = \frac{\alpha_4}{\epsilon}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are linear functions of y_1, y_2, y_3, y_4 and ϵ is a constant. If the coordinate lines form a conjugate system $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are solutions of (A) and if an asymptotic system, of (B). Substituting

$$\theta' = \frac{\theta}{\epsilon}$$

(A) is transformed into a system of the same kind in the first case, and the same is true of (B) in the second case. The theorem is thus proved.

Inversion preserves the lines of curvature. This transformation is given by

$y_1 = \frac{k^2 y_1'}{\rho}, \; y_2 = \frac{k^2 y_2'}{\rho}, \; y_3 = \frac{k^2 y_3'}{\rho}, \; y_4' = \frac{k^2 y_4}{\rho}$

where $\rho = y_1^2 + y_2^2 + y_3^2 + y_4^2$

Refer the hypersurface to its lines of curvature ρ_1, ρ_2, ρ_3. Then y_1, y_2, y_3, y_4, ρ satisfy the three simultaneous equations

$$\frac{\partial^2 \theta}{\partial \rho_i \partial \rho_j} = a_{ij} \frac{\partial \theta}{\partial \rho_1} + b_{ij} \frac{\partial \theta}{\partial \rho_2} + c_{ij} \frac{\partial \theta}{\partial \rho_3} \qquad i, j = 1, 2, 3 \quad i \neq j \qquad (c)$$

Now y_1, y_2, y_3, y_4 can be expressed in terms of y_1', y_2', y_3', y_4' as follows

$y_1 = \frac{k^2 y_1'}{\rho'}, \; y_2 = \frac{k^2 y_2'}{\rho'}, \; y_3 = \frac{k^2 y_3'}{\rho'}, \; y_4 = \frac{k^2 y_4'}{\rho'}$

where $\rho' = y_1'^2 + y_2'^2 + y_3'^2 + y_4'^2$ and $\rho = \frac{k^4}{\rho'}$

Effect on equations (c) the substitution

$$\theta = \frac{\sigma}{\rho'}$$

The equations in σ admit as particular solutions y_1', y_2', y_3', y_4' and 1, and are therefore of the form

$$\frac{\partial^2 \sigma}{\partial \rho_i \partial \rho_j} = a_{ij}' \frac{\partial \sigma}{\partial \rho_1} + b_{ij}' \frac{\partial \sigma}{\partial \rho_2} + c_{ij}' \frac{\partial \sigma}{\partial \rho_3} \qquad i, j = 1, 2, 3 \quad i \neq j \qquad (c')$$

For (c) admits the solution $\theta = 1$.

and hence (2) admits the solution x_4.
The hypersurface covers of the point
$M(y_1, y_2, y_3, y_4)$ has then x_1, x_2, x_3 as parameter ridge lines of curvature
and these are therefore preserved.
Consider now the three simultaneous equations —

$$\frac{\partial^2 \theta}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 \theta}{\partial x_1 \partial x_3} = 0 \quad \frac{\partial^2 \theta}{\partial x_2 \partial x_3} = 0$$

The general integral is of the form

$$\theta = f(x_1) + \varphi(x_2) + \psi(x_3)$$

where f, φ, ψ are arbitrary functions.
Writing

$$y_i = f_i(x_1) + \varphi_i(x_2) + \psi_i(x_3) \qquad i = 1, 2, 3, 4$$

we have a <u>hypersurface of translation</u> referred to a conjugate system. This may be generated by moving any one of the parametric surfaces so as to have its points describe curves congruent to the intersection of the other two.

The differential equation of the surface on which the asymptotic lines through a point lie, when the hypersurface is referred to a conjugate system, becomes

$$\Omega_{11}\, dx_1^2 + \Omega_{22}\, dx_2^2 + \Omega_{33}\, dx_3^2 = 0$$

and for a hypersurface of translation this takes the form,

$$\begin{vmatrix} f_1'' & f_2'' & f_3'' & f_4'' \\ f_1' & f_2' & f_3' & f_4' \\ \varphi_1' & \varphi_2' & \varphi_3' & \varphi_4' \\ \psi_1' & \psi_2' & \psi_3' & \psi_4' \end{vmatrix} dx_1^2 + \begin{vmatrix} \varphi_1'' & \varphi_2'' & \varphi_3'' & \varphi_4'' \\ f_1' & f_2' & f_3' & f_4' \\ \varphi_1' & \varphi_2' & \varphi_3' & \varphi_4' \\ \psi_1' & \psi_2' & \psi_3' & \psi_4' \end{vmatrix} dx_2^2 + \begin{vmatrix} \psi_1'' & \psi_2'' & \psi_3'' & \psi_4'' \\ f_1' & f_2' & f_3' & f_4' \\ \varphi_1' & \varphi_2' & \varphi_3' & \varphi_4' \\ \psi_1' & \psi_2' & \psi_3' & \psi_4' \end{vmatrix} dx_3^2 = 0$$

If now $f_1' = f_2' = \varphi_1' = \varphi_3' = \psi_2' = \psi_3' = 0$ this reduces at once to

$$\psi_1' \varphi_2' \begin{vmatrix} f_3'' & f_4'' \\ f_3' & f_4' \end{vmatrix} dx_1^2 + \psi_1' f_3' \begin{vmatrix} \varphi_2'' & \varphi_4'' \\ \varphi_2' & \varphi_4' \end{vmatrix} dx_2^2 + \varphi_2' f_3' \begin{vmatrix} \psi_4'' & \psi_1'' \\ \psi_4' & \psi_1' \end{vmatrix} dx_3^2 = 0$$

in which the variables can obviously be separated. Hence

The asymptotic cones are obtained by quadrature in all hypersurfaces of

nslation whose generating points lie in three mutually orthogonal hyperplanes.

The equations of the normal at any point on the hypersurface are

$$\sum(y'-y)\frac{\partial y}{\partial x_1}=0 \quad \sum(y'-y)\frac{\partial y}{\partial x_2}=0 \quad \sum(y'-y)\frac{\partial y}{\partial x_3}=0$$

or

$$\left.\begin{aligned}\sum y'\frac{\partial y}{\partial x_1}-\frac{\partial r}{\partial x_1}&=0\\ \sum y'\frac{\partial y}{\partial x_2}-\frac{\partial r}{\partial x_2}&=0\\ \sum y'\frac{\partial y}{\partial x_3}-\frac{\partial r}{\partial x_3}&=0\end{aligned}\right\} \qquad (1)$$

where $r=\sum\frac{y^2}{2}$

Expressing the condition that the normal trace out a developable surface we have for a point y' suitably chosen on the normal

$$\sum\frac{\partial y}{\partial x_1}dy'=0 \quad \sum\frac{\partial y}{\partial x_2}dy'=0 \quad \sum\frac{\partial y}{\partial x_3}dy'=0$$

Differentiating (1) under this hypothesis

$$\left.\begin{aligned}\sum y'\,d\frac{\partial y}{\partial x_1}-d\frac{\partial r}{\partial x_1}&=0\\ \sum y'\,d\frac{\partial y}{\partial x_2}-d\frac{\partial r}{\partial x_2}&=0\\ \sum y'\,d\frac{\partial y}{\partial x_3}-d\frac{\partial r}{\partial x_3}&=0\end{aligned}\right\} \qquad (2)$$

Eliminating y_1', y_2', y_3', y_4' between (1) and (2) we have the five equations

$$\begin{vmatrix} \lambda\frac{\partial y_1}{\partial x_i} & \lambda\frac{\partial y_1}{\partial x_j} & \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \lambda\frac{\partial y_2}{\partial x_i} & - & - & - & - \\ \lambda\frac{\partial y_3}{\partial x_i} & - & - & - & - \\ & - & - & - & - \\ \lambda\frac{\partial y_4}{\partial x_i} & - & - & - & - \\ \lambda\frac{\partial r}{\partial x_i} & - & - & - & - \end{vmatrix} = 0 \qquad \begin{matrix} i,j = 1\,2\,3 \\ i \neq j \end{matrix}$$

which give the lines of curvature.

Consider now the three simultaneous equations

$$a'_{ij}\frac{\partial^2\theta}{\partial x_i^2} + b'_{ij}\frac{\partial^2\theta}{\partial x_i\partial x_j} + c'_{ij}\frac{\partial^2\theta}{\partial x_j^2} + a_{ij}\frac{\partial\theta}{\partial x_1} + b_{ij}\frac{\partial\theta}{\partial x_2} + c_{ij}\frac{\partial\theta}{\partial x_3} = 0 \qquad (4)$$

$$i \neq j, \qquad i,j = 1, 2, 3,$$

and write the conditions that these admit the five particular solutions y_1, y_2, y_3, y_4, r. Equations (4) can then be written in the determinant form

$$\begin{vmatrix} \frac{\partial^2\theta}{\partial x_i^2} & \frac{\partial^2\theta}{\partial x_i\partial x_j} & \frac{\partial^2\theta}{\partial x_j^2} & \frac{\partial\theta}{\partial x_1} & \frac{\partial\theta}{\partial x_2} & \frac{\partial\theta}{\partial x_3} \\ \frac{\partial^2 y_1}{\partial x_i^2} & - & - & - & - & - \\ \vdots & & & & & \\ \frac{\partial^2 r}{\partial x_i^2} & - & - & - & - & - \end{vmatrix} = 0$$

Then using equations (5) with (3) we see that these can be written

$$C_{ij}\,dx_i^2 - B_{ij}\,dx_i\,dx_j - A_{ij}\,dx_i^2 + L_{ij}\,dx_i\,dx_k + M_{ij}\,dx_j\,dx_k + N_{ij}\,dx_k^2 = 0 \quad (6)$$

$$i, j, k = 1, 2, 3. \quad i \neq j \neq k$$

where

$$L_{ij} = \left|\frac{\partial^2 u}{\partial x_i^2} \;\; \frac{\partial^2 u}{\partial x_j \partial x_k}\right| + \left|\frac{\partial^2 u}{\partial x_i \partial x_k} \;\; \frac{\partial^2 u}{\partial x_i \partial x_j}\right|$$

$$M_{ij} = \left|\frac{\partial^2 u}{\partial x_i \partial x_j} \;\; \frac{\partial^2 u}{\partial x_i \partial x_k}\right| + \left|\frac{\partial^2 u}{\partial x_i \partial x_k} \;\; \frac{\partial^2 u}{\partial x_j^2}\right|$$

$$N_{ij} = \left|\frac{\partial^2 u}{\partial x_i \partial x_k} \;\; \frac{\partial^2 u}{\partial x_j \partial x_k}\right|$$

$$\left|\frac{\partial^2 u}{\partial x_i \partial x_j} \;\; \frac{\partial^2 u}{\partial x_i \partial x_k}\right| = \begin{vmatrix} \frac{\partial^2 u_1}{\partial x_i \partial x_j} & \frac{\partial^2 u_1}{\partial x_i \partial x_k} & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \vdots & & & & \\ \frac{\partial^2 v}{\partial x_i \partial x_j} & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

Now $\left|\frac{\partial^2 u}{\partial x_i \partial x_j} \;\; \frac{\partial^2 u}{\partial x_i \partial x_k}\right|$ can at once be expressed in terms of $\left|\frac{\partial^2 u}{\partial x_1^2} \;\; \frac{\partial^2 u}{\partial x_2^2}\right|$, $\left|\frac{\partial^2 u}{\partial x_1^2} \;\; \frac{\partial^2 u}{\partial x_3^2}\right|$, $\left|\frac{\partial^2 u}{\partial x_2^2} \;\; \frac{\partial^2 u}{\partial x_3^2}\right|$, A_{ij}, B_{ij}, C_{ij} by means of equation (3).

Since L_{ij}, M_{ij}, N_{ij} are immediately expressible in terms of A_{ij}, B_{ij}, C_{ij}. We see then that if by any means we can obtain three partial differential equations of the form (4) simultaneously satisfied by y_1, y_2, y_3, y_4, y_5, then the differential equations of the lines of curvature are given by (6).

If in particular $A_{ij} = C_{ij} = 0$ then obviously $L_{ij} = M_{ij} = N_{ij} = 0$ and (6) reduces to

$$dx_1\, dx_2 = 0 \quad dx_1\, dx_3 = 0 \quad dx_2\, dx_3 = 0$$

the hypersurface being referred to its lines of curvature as we have already seen. The transformation of variables defined by (6) must then reduce A_{ij} and C_{ij} simultaneously to zero, and (6) plays the same role with regard to (4) that the equation of characteristics does with respect to the

single equation

$$A\frac{\partial^2\theta}{\partial x^2} + B\frac{\partial^2\theta}{\partial x\,\partial p} + C\frac{\partial^2\theta}{\partial p^2} + A'\frac{\partial\theta}{\partial x} + B'\frac{\partial\theta}{\partial p} = 0$$

We may then say:

Whenever we have obtained three partial differential equations of the form (4) which are simultaneously satisfied by y_1, y_2, y_3, y_4 the differential equations of the conjugate lines are given by (6) and if in addition (4) admits $y_1^2+y_2^2+y_3^2+y_4^2$ as a particular solution (6) defines the lines of curvature.

We can now extend the definition of conjugate lines of the hypersurface.

Write the tangent hyperplane at $M(y_1\, y_2\, y_3\, y_4)$

$$Y_4 - y_4 = p_1(Y_1 - y_1) + p_2(Y_2 - y_2) + p_3(Y_3 - y_3) \qquad (7)$$

where $p_i = \frac{\partial y_4}{\partial x_i}$

Suppose this tangent hyperplane displaced along the three coordinate lines x_1, x_2, x_3 (

The characteristic in each case is a plane and these three planes are given by (7) and each of

$$\sum \frac{\partial f}{\partial x_1}(y'-y)=0 \quad \sum \frac{\partial f}{\partial x_2}(y'-y)=0 \quad \sum \frac{\partial f}{\partial x_3}(y'-y)=0 \qquad (8)$$

Expressing the conditions that the lines formed by taking these three planes in pairs are tangent to the coordinate lines. For this it is necessary and sufficient to write that the ~~three~~ six equations formed by (7) and each of (8) are satisfied when (y'_1-y_1), (y'_2-y_2), (y'_3-y_3) (y'_4-y_4) are replaced by $\frac{\partial y_1}{\partial x_i}$, $\frac{\partial y_2}{\partial x_i}$, $\frac{\partial y_3}{\partial x_i}$ $\frac{\partial y_4}{\partial x_i}$ respectively substituted.

This leads at once to

$$\frac{\partial^2 y_4}{\partial x_i \partial x_j} - p_1 \frac{\partial^2 y_1}{\partial x_i \partial x_j} - p_2 \frac{\partial^2 y_2}{\partial x_i \partial x_j} - p_3 \frac{\partial^2 y_3}{\partial x_i \partial x_j} = 0 \qquad (9)$$

$(i, j = 1, 2, 3 \quad i \neq j)$

We have also

$$\frac{\partial y_4}{\partial x_i} = p_1 \frac{\partial y_1}{\partial x_i} + p_2 \frac{\partial y_2}{\partial x_i} + p_3 \frac{\partial y_3}{\partial x_i}$$

Eliminating p_1, p_2, p_3 between (9 and 10) taken in pairs we get as the required conditions

$$\begin{vmatrix} \dfrac{\partial^2 y_1}{\partial x_i \partial x_j} & \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial^2 y_2}{\partial x_i \partial x_j} & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 y_3}{\partial x_i \partial x_j} & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 y_4}{\partial x_i \partial x_j} & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad \begin{matrix} (i, j = 1, 2, 3) \\ (i \neq j) \end{matrix}$$

These are the conditions that the four punctual coordinates of the hypersurface satisfy three simultaneous partial differential equations of the type

$$\frac{\partial^2 \theta}{\partial x_i \partial x_j} = a_i \cdot \frac{\partial \theta}{\partial x_1} + b_j \frac{\partial \theta}{\partial x_2} + c_i \frac{\partial \theta}{\partial x_3} \qquad (12)$$
$$(i, j = 1, 2, 3. \quad i \neq j)$$

Hence we may define as conjugate lines on the hypersurface lines such that the tangent

hyperplane is displaced along any two of them, the intersection of the characteristics is tangent to the third.

The equations satisfied by the tangential coordinates can also be easily derived by following the reasoning of Darboux* for the case of three dimensions.

Writing y_1, y_2, y_3, y_4, y_5 the homogeneous coordinates of a point the tangent hyperplane is

$$u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 + u_5 y_5 = 0 \qquad (13)$$

where u_i is a function of $\beta_1, \beta_2, \beta_3$ the independent parameters. The hypersurface is obtained by eliminating $\beta_1, \beta_2, \beta_3$ between (13) and its three partial derivatives

$$\sum y \frac{\partial u}{\partial \beta_1} = 0 \quad \sum y \frac{\partial u}{\partial \beta_2} = 0 \quad \sum y \frac{\partial u}{\partial \beta_3} = 0 \qquad (14)$$

* Théorie Générale des Surfaces [illegible]

Since $y(\rho_1, \rho_2, \rho_3)$ is the point of contact of the hyper-tangent plane, then

$$\sum u y = 0, \quad \sum y \frac{\partial u}{\partial \rho_1} = 0, \quad \sum y \frac{\partial u}{\partial \rho_2} = 0, \quad \sum y \frac{\partial u}{\partial \rho_3} = 0 \qquad (15)$$

which lead at once to

$$\sum u \frac{\partial y}{\partial \rho_1} = 0, \quad \sum u \frac{\partial y}{\partial \rho_2} = 0, \quad \sum u \frac{\partial y}{\partial \rho_3} = 0 \qquad (16)$$

Noting now the conditions that ρ_1, ρ_2, ρ_3 form a conjugate system we have at once

$$\sum_k u_k \frac{\partial^2 y_k}{\partial \rho_i \partial \rho_j} = 0, \quad \sum_k y_k \frac{\partial^2 u_k}{\partial \rho_i \partial \rho_j} = 0 \quad \begin{matrix} (i, j = 1, 2, 3) \\ (i \neq j) \end{matrix} \qquad (17)$$

Eliminating $y_1 \ldots y_5$ between (15) and the first three of (17) we get the equations satisfied by the tangential coordinates

$$\begin{vmatrix} \frac{\partial^2 u_1}{\partial \rho_i \partial \rho_j} & \frac{\partial u_1}{\partial \rho_1} & \frac{\partial u_1}{\partial \rho_2} & \frac{\partial u_1}{\partial \rho_3} & u_1 \\ \vdots & & & & \\ \frac{\partial^2 u_5}{\partial \rho_i \partial \rho_j} & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \quad \begin{matrix} (i, j = 1, 2, 3) \\ (i \neq j) \end{matrix} \qquad (18)$$

with three similar ones for the point coordinates. Hence

Given the three partial differential equations

$$A_i \frac{\partial^2\theta}{\partial y_1^2} + B_i \frac{\partial^2\theta}{\partial y_1 \partial y_2} + C_i \frac{\partial^2\theta}{\partial y_2^2} + D_i \frac{\partial\theta}{\partial y_1} + E_i \frac{\partial\theta}{\partial y_2} + G_i \frac{\partial\theta}{\partial y_3} + H_i\theta = 0 \quad (19)$$

which are satisfied by the homogeneous cartesian or tangential coordinates of a surface, suppose the system of equations

$$C_i\,dy_1^2 - B_i\,dy_1 dy_2 + A_i\,dy_2^2 + E_i\,dy_2 dy_3 - H_i\,dy_1 dy_3 + F_i\,dy_3^2 = 0 \quad (20)$$

gives the conjugate lines, and it is easily seen that if (19) admits $y_1^2+y_2^2+y_3^2+y_4^2$ as a particular solution in the first case and $u_1^2+u_2^2+u_3^2-u_4^2$ in the second case, it also gives the lines of curvature.

V

On the Spherical Representation

If after the manner of the spherical representation of Gauss we suppose given a radius from the centre of the hypersphere with

sent the origin so as to be parallel to the normals of the hypersurface along any line these will trace out on the hypersphere the spherical representation of the line on the hypersurface. Write the corresponding linear element on the hypersphere

$$ds'^2 = e_{11}dt^2 + e_{22}du^2 + e_{33}dv^2 + 2e_{12}\,dt\,du + 2e_{13}\,dt\,dv + 2e_{23}\,du\,dv \qquad (1)$$

We have then at once from (II) page 19

$$\left.\begin{aligned}
e_{11} &= \Sigma\left(\frac{\partial X}{\partial t}\right)^2 = \frac{1}{\Delta}\left(\partial_{11}|\partial_{11}^1| + \partial_{21}|\partial_{11}^2| + \partial_{31}|\partial_{11}^3|\right)\\
e_{22} &= \Sigma\left(\frac{\partial X}{\partial u}\right)^2 = \frac{1}{\Delta}\left(\partial_{12}|\partial_{12}^1| + \partial_{22}|\partial_{12}^2| + \partial_{32}|\partial_{12}^3|\right)\\
e_{33} &= \Sigma\left(\frac{\partial X}{\partial v}\right)^2 = \frac{1}{\Delta}\left(\partial_{13}|\partial_{13}^1| + \partial_{23}|\partial_{13}^2| + \partial_{33}|\partial_{13}^3|\right)\\
e_{12} &= \Sigma\frac{\partial X}{\partial t}\frac{\partial X}{\partial u} = \frac{1}{\Delta}\left(\partial_{12}|\partial_{11}^1| + \partial_{22}|\partial_{11}^2| + \partial_{32}|\partial_{11}^3|\right)\\
&= \frac{1}{\Delta}\left(\partial_{11}|\partial_{12}^1| + \partial_{21}|\partial_{12}^2| + \partial_{31}|\partial_{11}^3|\right)\\
e_{13} &= \Sigma\frac{\partial X}{\partial t}\frac{\partial X}{\partial v} = \frac{1}{\Delta}\left(\partial_{13}|\partial_{11}^1| + \partial_{23}|\partial_{11}^2| + \partial_{33}|\partial_{11}^3|\right)\\
&= \frac{1}{\Delta}\left(\partial_{11}|\partial_{13}^1| + \partial_{21}|\partial_{13}^2| + \partial_{31}|\partial_{13}^3|\right)\\
e_{23} &= \Sigma\frac{\partial X}{\partial u}\frac{\partial X}{\partial v} = \frac{1}{\Delta}\left(\partial_{13}|\partial_{12}^1| + \partial_{23}|\partial_{12}^2| + \partial_{33}|\partial_{12}^3|\right)\\
&= \frac{1}{\Delta}\left(\partial_{12}|\partial_{13}^1| + \partial_{22}|\partial_{13}^2| + \partial_{32}|\partial_{13}^3|\right)
\end{aligned}\right\} \qquad (2)$$

From these it is easily seen that the fully orthogonal system formed by the lines of curvature on the hypersurface goes into a fully orthogonal system on the hypersphere for we having

$$\mathcal{E}_{12} = \mathcal{E}_{13} = \mathcal{E}_{23} = \vartheta_{12} = \vartheta_{13} = \vartheta_{23} = 0$$

we have at once

$$e_{12} = e_{13} = e_{23} = 0$$

Let us now see what relations exist between the angles made by a set of conjugate lines on the hypersurface and those made by their spherical representations.

In this case

$$\vartheta_{12} = \vartheta_{13} = \vartheta_{23} = 0$$

$$e_{11} = \frac{\vartheta_{11}^2}{4}\begin{vmatrix}\mathcal{E}_{22} & \mathcal{E}_{23} \\ \mathcal{E}_{32} & \mathcal{E}_{33}\end{vmatrix} \equiv \frac{\vartheta_{11}^2}{4}\mathcal{E}_{11}' \quad \text{say}.$$

$$e_{22} = \frac{\vartheta_{22}^2}{4}\mathcal{E}_{22}' \qquad e_{33} = \frac{\vartheta_{33}^2}{4}\mathcal{E}_{33}'$$

$$e_{12} = \frac{\vartheta_{11}\vartheta_{22}}{4}\mathcal{E}_{12}', \quad e_{13} = \frac{\vartheta_{11}\vartheta_{33}}{4}\mathcal{E}_{13}', \quad e_{23} = \frac{\vartheta_{22}\vartheta_{33}}{4}\mathcal{E}_{23}'$$

Let Ω_{ij} have the same significance in the hyperplane as ω_{ij} in the hypersurface. Then

$$\cos \Omega_{12} = \frac{e_{12}}{\sqrt{e_{11} e_{22}}} = \frac{\mathcal{E}'_{12}}{\sqrt{\mathcal{E}'_{11}\mathcal{E}'_{22}}}$$

From

$$\mathcal{E}_{12} = \sqrt{\mathcal{E}_{11}\mathcal{E}_{22}} \cos \omega_{12}$$
$$\mathcal{E}_{13} = \sqrt{\mathcal{E}_{11}\mathcal{E}_{33}} \cos \omega_{13}$$
$$\mathcal{E}_{23} = \sqrt{\mathcal{E}_{22}\mathcal{E}_{33}} \cos \omega_{23}$$

we have at once

$$\mathcal{E}'_{12} = \begin{vmatrix} \sqrt{\mathcal{E}_{11}\mathcal{E}_{22}} \cos \omega_{12} & \sqrt{\mathcal{E}_{22}\mathcal{E}_{33}} \cos \omega_{23} \\ \sqrt{\mathcal{E}_{11}\mathcal{E}_{33}} \cos \omega_{1} & \mathcal{E}_{33} \end{vmatrix} = \mathcal{E}_{33}\sqrt{\mathcal{E}_{11}\mathcal{E}_{22}} \begin{vmatrix} \cos \omega_{12} & \cos \omega_{23} \\ \cos \omega_{13} & 1 \end{vmatrix}$$

$$\mathcal{E}'_{11} = \begin{vmatrix} 1 & \cos \omega_{23} \\ \cos \omega_{23} & 1 \end{vmatrix} \mathcal{E}_{22}\mathcal{E}_{33} = \mathcal{E}_{22}\mathcal{E}_{33} \sin^2 \omega_{23}$$

which give

$$\cos \Omega_{12} = \frac{\cos \omega_{12} - \cos \omega_{13} \cos \omega_{23}}{\sin \omega_{13} \sin \omega_{23}}$$

$$\cos \Omega_{13} = \frac{\cos \omega_{13} - \cos \omega_{12} \cos \omega_{23}}{\sin \omega_{12} \sin \omega_{23}}$$

$$\cos \Omega_{23} = \frac{\cos \omega_{23} - \cos \omega_{12} \cos \omega_{13}}{\sin \omega_{12} \sin \omega_{13}}$$

If two of the conjugate lines of a system in a surface of curvature the third is orthogonal to them hence

The angle between two conjugate lines lying in a surface of curvature of any hypersurface is therefore changed into its supplement by spherical representation.

Let ω be the solid angle formed by any three parametric surfaces at a point upon the hypersurface. The element of hyperarea at this time

$$dS = \sin\omega \, ds_1 \, ds_2 \, ds_3$$

and the corresponding element of hyperarea on the hypersphere is

$$dS' = \sin\Omega \, ds_1' \, ds_2' \, ds_3'$$

It is easily seen that

$$\frac{dS'}{dS} = K$$

the total curvature of the hypersurface.

[illegible]

$$\sin^2 \omega = \begin{vmatrix} 1 & \cos\omega_{12} & \cos\omega_{13} \\ \cos\omega_{12} & 1 & \cos\omega_{23} \\ \cos\omega_{13} & \cos\omega_{23} & 1 \end{vmatrix} = \varepsilon_{11}\varepsilon_{22}\varepsilon_{33}\,\Delta$$

$$\sin^2 \Omega = \begin{vmatrix} 1 & \cos\Omega_{12} & \cos\Omega_{13} \\ \cos\Omega_{12} & 1 & \cos\Omega_{23} \\ \cos\Omega_{13} & \cos\Omega_{23} & 1 \end{vmatrix} = e_{11}e_{22}e_{33}\,\Gamma$$

where $\quad \Gamma = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix}$

Again $ds_t\, ds_u\, ds_r = \dfrac{1}{\sqrt{\varepsilon_{11}\varepsilon_{22}\varepsilon_{33}}}\, dt\, du\, dv$

$ds'_t\, ds'_u\, ds'_r = \dfrac{1}{\sqrt{e_{11}e_{22}e_{33}}}\, dt\, du\, dv$

Hence $\quad \dfrac{dS'}{dS} = \sqrt{\dfrac{\Gamma}{\Delta}}$

Now it is easily seen that

$$\sqrt{\Gamma} = \begin{vmatrix} \frac{\partial X}{\partial t} & \frac{\partial Y}{\partial t} & \frac{\partial Z}{\partial t} & \frac{\partial W}{\partial t} \\ \frac{\partial X}{\partial u} & - & - & - \\ \frac{\partial X}{\partial v} & - & - & - \\ X & - & - & - \end{vmatrix} \qquad \sqrt{\Delta} = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} & \frac{\partial w}{\partial t} \\ \frac{\partial x}{\partial u} & - & - & - \\ \frac{\partial x}{\partial v} & - & - & - \\ X & - & - & - \end{vmatrix}$$

from which it follows [illegible]

$$D = -\sqrt{\delta\Delta}$$

or $K = -\dfrac{D}{4} = \sqrt{\dfrac{\delta}{\Delta}}$

Hence $\qquad \dfrac{dS'}{dS} = K$

We have the test:

The limit of the ratio of the element of hyperarea on the hypersphere to the corresponding element of hyperarea on the hypersurface is equal to the total curvature of the hypersurface.

Consider now the following problem. Given the third differential quadratic form

$$ds'^2 = \sum_{1}^{3} e_{ij}\, dx_i\, dx_j \qquad (3)$$

where the e_{ij} are assigned functions of x_1, x_2, x_3; required the corresponding hypersurface.

Applying equations II and (4)

page 19. The hypothesis we have

$$\frac{\partial^2 \Theta}{\partial x_i \partial x_j} = \sum_k^3 \left\{ {ij \atop k} \right\}' \frac{\partial \Theta}{\partial x_k} - e_{ij}\Theta \quad \begin{pmatrix} i,j=1,2,3, \\ i \neq j \end{pmatrix} \qquad (4)$$

where $\left\{ {ij \atop k} \right\}'$ is constructed with respect to (3).

For excluding the hypersurface of zero curvature, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ can be so determined that the four equations

$$f_i = \alpha_1 \frac{\partial \Theta_i}{\partial x_1} + \alpha_2 \frac{\partial \Theta_i}{\partial x_2} + \alpha_3 \frac{\partial \Theta_i}{\partial x_3} + \alpha_4 \Theta_i \quad (i=1,2,3,4) \qquad (5)$$

shall be satisfied where f_1, f_2, f_3, f_4 are any four functions of x_1, x_2, x_3, since the determinant of the system does not vanish being equal to the square root of the discriminant of (3). Putting $f_i = \frac{\partial \varphi_i}{\partial x_1}$ and multiplying the four equations (5) in order by $\frac{\partial \Theta_i}{\partial x_1}, \frac{\partial \Theta_i}{\partial x_2}, \frac{\partial \Theta_i}{\partial x_3}, \Theta_i$ and summing in each case, we have

$$-D_{k1} = \sum_j^3 \alpha_j \cdot e_{kj} \qquad 0 = \alpha_4$$
$$(k=1,2,3)$$

solving these

$$\sigma_k = -\frac{1}{\delta}|D_{11}^k|' \qquad (k = 1, 2, 3)$$

where in $|D_{ii}^k|'$ ε_{im} is replaced by e_{im}

We have thus at once the equations

$$\frac{\partial\theta}{\partial x_i} = -\frac{1}{\delta}\left(|D_{i1}^1|'\frac{\partial\Theta}{\partial x_1} + |D_{i1}^2|'\frac{\partial\Theta}{\partial x_2} + |D_{i1}^3|'\frac{\partial\Theta}{\partial x_3}\right) \qquad (6)$$

$$(i = 1, 2, 3)$$

These three equations are of exactly the same form as (II) page 19 with Θ written for θ and e_{ij} for ε_{ij} and therefore the conditions of integrability of (6) are constructed with respect to (3) in exactly the same way that the conditions of integrability for (II) were constructed with respect to the form

$$\varphi^2 = \sum \varepsilon_{ij}\, dx_i\, dx_j$$

Referring then to equations (II) page 19 we have at once as the conditions of integrability of (6),

$$\frac{\partial b_{rs}}{\partial x_t} - \frac{\partial b_{st}}{\partial x_r} + \sum_{i=1}^{3}\left(b_{it}\left\{\begin{matrix} rs \\ i \end{matrix}\right\}' - b_{ir}\left\{\begin{matrix} st \\ i \end{matrix}\right\}'\right) \qquad (7)$$

$$(r, s, t = 1, 2, 3; \quad t \neq r)$$

hence we may say

Given the two differential quadratic forms

$$f = \sum_{1}^{3} a_{rs}\, dx_r\, dx_s$$

$$f' = \sum_{1}^{3} b_{rs}\, dx_r\, dx_s$$

of which the first is definite and of curvature +1 (belongs to the hypersphere), then in order that there exist a hypersurface admitting these as second and third fundamental forms it is necessary and sufficient that equations (7) be satisfied and the corresponding hypersurface is obtained by the integration of (6).

Vita

George Oscar James was born near Norfolk Virginia August 1 1873. He received his elementary education at Churchland Academy near Norfolk and entered the Johns Hopkins University in October of 1892 as candidate for the degree of Bachelor of Arts in the mathematical-physical group. In June of 1895 he was graduated and appointed University Scholar in Mathematics selecting Physics and Astronomy as first and second subordinates. In October of 1896 he resigned the appointment of Honorary Hopkins Scholar, becoming Instructor in Physics in the University of Utah and [illegible]

entered the Johns Hopkins University in October of 1877 In June of 1878 he was made Fellow in Mathematics.